How Should Examples be Learned in a Production Task? An Experimental Investigation in Mathematical Problem Posing

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Abstract

When using mathematics in problem solving in everyday life, problem solvers must recognize and formulate problems by themselves because structured problems are not provided. Therefore, it is an important task in general education to foster learner problem posing. Although learning by solving examples is adopted in general education, it may not be sufficiently effective in fostering learner problem posing because cognitive skills differ between problem solving and problem posing. This study experimentally investigated the effects of three learning activities in problem posing: learning by solving an example, learning by reproducing an example, and learning by evaluating an example. In our experiment, undergraduates were asked to pose their own new and unique problems from a base problem initially given after learning an example by solving, reproducing, or evaluating it. The results indicated that learning by reproducing the example was the most effective in fostering the composition of new problems.

Keywords: Production task; problem posing; learning from examples; mathematical learning.

Introduction

In addition to solving problems given by a teacher or textbook, problem posing, by which learners create problems, has also been identified as an important activity in mathematics education. In fact, some mathematicians and mathematics educators have pointed out that problem posing lies at the heart of mathematical activity (Polya, 1945; Silver, 1994). Problem posing is a necessary skill for problem solving in everyday life. Because structured problems are not provided when using mathematics in everyday life, problem solvers must recognize and formulate problems by themselves (Ishida & Inoue, 1983; Singer & Voica, 2013). Therefore, it is an important task in general education to foster learner problem posing. Several studies have addressed this issue in terms of a learning activity to improve problem solving, despite insufficiently addressing the skill of learner problem posing itself.

In the research of problem posing, it was empirically confirmed that novice learners succeeded in posing new problems based on mathematical relationships given in formulae or equations (we refer to these as *solutions*), whereas they had difficulty in posing problems by composing novel solutions on their own (Christou et al., 2005; Kojima, Miwa, & Matsui, 2010). Because problem posing in everyday life is performed under various constraints by using different materials, it is desirable to foster skill to pose diverse problems appropriately. Thus, a learning method to improve the composition of solutions by novice learners is required.

To support learning by novice learners, it is efficient and effective to use examples. Examples are indispensable in learning in any domain, including mathematics. In general mathematics education, procedures to reach answers in solving problems are initially instructed with examples. However, the general method of learning from examples in problem solving may not be sufficiently effective in problem posing because cognitive skills in problem solving and problem posing are different. We refer to the former task as a *comprehension task*, and the latter as a *production task*. In fact, it is reported that learning different tasks such as comprehension and production has no influence on the other (Singley & Anderson, 1989). In comprehension tasks, transfer of a solution learned in an example to problem solving has been argued (e.g., Gick & Holyoak, 1983; Novick & Holyoak, 1991) and adaptive scaffolding that enhances learning from examples has been discussed (e.g., Conati & VanLehn, 2000; Schwonke, Renkl, Krieg, Wittwer, Aleven, & Salden, 2009; McLaren & Isotani, 2011). However, the central issue is basically limited to problem solving and does not include problem posing.

This study experimentally investigated the effects of activities for learning from an example in problem posing. In our experiment, undergraduates were asked to pose their own new and unique problems from a base problem initially given after they had learned an example. We compared three activities for learning from an example: learning by solving an example used in general education, learning by reproducing an example and learning by evaluating an example. The third activity of evaluation is adopted in some studies (Hirai, Hazeyama & Inoue, 2009; Takagi & Teshigawara, 2006; Yu, Liu & Chan, 2005), in which a learner evaluates problems posed by other learners. However, the focus of these studies was mainly on improving comprehension of declarative knowledge through the evaluation of problems, not on the production of problems itself. The second activity of reproduction was proposed in our previous study (Kojima, Miwa & Matsui, 2013), in which a learner understands an example from the viewpoint of the poser by reproducing the same problem as the example.

Methods

Procedures and Materials

Undergraduates participated in the experimental investigation conducted in three classes of a cognitive science lecture from 2010 to 2012. They were engaged in two tasks, each requiring the posing of new problems from a one initially given as a base.

The undergraduates were first given a learning task in the domain of word problems solved with simultaneous equations. They were told that the purpose of the learning task was to instruct them how to pose a novel problem from a base. The base in the learning task was the following problem A_1.

A_1: I bought some 60-yen oranges and 120-yen apples for 1020 yen. The total number of oranges and apples was 12. How many oranges and apples did I buy?
Solution:
Let x denote the number of oranges and y denote the number of apples.
$x + y = 12$
$60x + 120y = 1020$
According to the equations above, $x = 7$ and $y = 5$.

The undergraduates learned the following problem A_2 as an example of output in the domain of A_1.

A_2: Last year, I bought some 40-yen pencils and 110-yen pens. The total number was 13. This year, I bought 2 times as many pencils as last year, as many pens as last year, and a 300-yen pen case for 1430 yen. How many pencils and pens did I buy last year?
Solution.
Let x denote the number of pencils and y denote the number of pens.
$x + y = 13$
$40 \times 2x + 110y = 1430 - 300$
According to the equations above, $x = 10$ and $y = 3$.

The solution of A_2 was composed by an alteration that added two parameters and operations to A_1. Thus, it can be a hint for composing novel solutions in problem posing by the undergraduates.

The learning task was followed by a problem posing task, in which the undergraduates were asked to pose their own problems in the domain of word problems solved with unitary equations. The base in the problem posing task was the following problem B.

B: I want to buy some boxes of cookies. If I buy 110-yen boxes of cookies, then I have 50 yen left. If I buy 120-yen boxes of chocolate cookies, then I need 20 yen more. How many boxes do I want?
Solution.
Let x denote the number of boxes.
$110x + 50 = 120x - 20$
According to the above equation, $x = 7$.

Prior to the start of the problem posing task, the undergraduates were instructed to pose as many, diverse and unique problems as possible in 20 minutes.

Condition Groups

In the 2010 class, undergraduates were provided sheets of paper on which the text and solution of A_1 and the text of A_2 were printed in the learning task. They were asked to solve A_2 and write the answer in the sheet. Thus, we refer to the undergraduates as a *solving group*.

In 2011, undergraduates were first presented A_1 and A_2 on a screen in front of the classroom. They were then provided sheets on which A_1 and information indicating how to compose A_2 from A_1 were printed after A_2 had been removed from the screen. They were asked to reproduce the same problem as A_2 according to the information. They were also told that texts of their problems did not need to be identical to the example as long as it could be solved by the solution identical to the example. The purpose of using such information, not the example itself, is to prevent mere duplication of the characters and symbols composing the example. (For details on reproducing information of the composition of an example, see Kojima et al., 2013.) We refer to the undergraduates as a *reproduction group*. Appendix A shows the information presented to this group.

In 2012, undergraduates were provided sheets on which A_1 and A_2 were printed. They were asked to evaluate A_2 from the viewpoints of the *originality* and *feasibility* as a mathematical problem by using 5 point scales and describe the reasons of the evaluations. These viewpoints are generally used in the research of creative thinking. We refer to these undergraduates as an *evaluation group*.

Analysis

Problems posed by the undergraduates in the problem posing task were analyzed in terms of the variety, strategies to alter solutions, and the complexities of solutions. The variety of each problem was evaluated based on the four categories shown in Figure 1, which indicate similarities in *situations* and solutions between the problem and base. Situations of problems denote contextual settings expressed in texts such as *purchase of goods* or *transfer by vehicles*. Category I/I indicates problems that are almost the same as the base; D/I indicates problems generated by altering the situations of the base; I/D indicates problems generated by altering the solutions; and D/D indicates problems generated by combining alterations in both situations and solutions. The example A_2 is a problem in Category I/D.

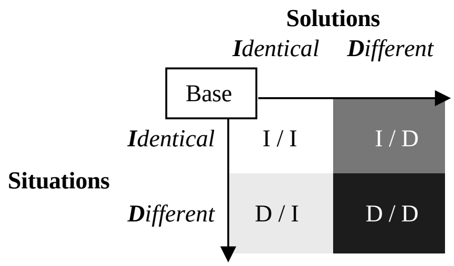

Figure 1: Categories of problems for evaluating varieties.

Strategies to alter solutions of problems posed by the undergraduates were evaluated by comparing the solution structure of each problem with that of the base. The undergraduates' problems were classified into *not altered*, *partially altered* (adding/removing operations to/from the solution of the base) or *overall altered* (composing a solution entirely different from the base). A_2 was posed with partially altered.

The complexities of the undergraduates' problems were estimated by comparing the number of operations required to reach the answers in each problem with that of base. The number of operations in the base is three. Only the complexities of I/D and D/D problems were analyzed because the structure of solutions in I/I and D/I problems are always equal to the base. These classifications of the undergraduates' problems were conducted by the first author according to features of the problem texts and formal structures of solutions.

We had acquired problems posed by undergraduates in the same problem posing task without learning of any example in the previous study (Kojima et al., 2010). Results indicated that the undergraduates successfully posed many I/I and D/I problems but posed few I/D problems, and their I/D and D/D problems mostly had simple and inappropriate solutions. The effects of learning the example were verified through comparison of the solving, reproduction, and evaluation groups in this experiment (*experimental groups*) with the previous study as a *control group*.

In the reproduction group, some undergraduates did not reproduce A_2 but posed problems different from A_2 in some ways (e.g., changing parameters or operations in A_2), and some did not complete reproduction in the learning task. Such undergraduates were excluded from the analysis. Some of others in the reproduction group failed in reproducing A_2. Although they wrote the same solution to A_2, their problem texts were contradictory to the solution. Therefore, data of those who failed in the learning task (*reproduction-f group*) were separately described from the others who succeeded (*reproduction-s group*).

In the solving group, 62 undergraduates participated; in the reproduction group, 132; and in the evaluation group, 25. In the reproduction group, 44 did not reproduce A_2, and 8 did not complete reproduction. In the others, 52 were in the reproduction-s group, and 28 were in the reproduction-f group. The numbers of participants in each group differed because the numbers of undergraduates in the lecture classes each year varied. In the control group, 76 undergraduates participated.

Results

Undergraduates in the solving, reproduction-s, reproduction-f and evaluation groups posed 372 problems in the problem posing task. Sixty eight of the posed problems were excluded because they were in domains other than the base (e.g., solved with inequalities) or unsolvable due to insufficient or contradictory constraints. Appendix B shows some examples of problems posed in the experimental groups. In the same way, the control group posed 146 problems and 29 of them were excluded.

The varieties

Figure 2 indicates the proportions of posed problems in each category. As mentioned above, the control group posed few I/D problems. The experimental groups posed more I/D problems than the control group. We compared the control group with the solving group using the chi-square test; the result indicated a significant difference between the solving and control groups ($\chi^2(3) = 11.51$, $p < .01$). Furthermore, the results of residual analysis indicated that the number of D/I problems in the control group was significantly high but significantly low in the solving group. The number of I/D problems in the solving group was significantly high but significantly low in the control group. Similarly, a significant difference existed between the reproduction-s and control groups ($\chi^2(3) = 15.26$, $p < .01$). The number of I/I problems in the control group was significantly high but significantly low in the reproduction-s group. The number of I/D problems in the reproduction-s group was significantly high but significantly low in the control group. There was also a significant difference between the evaluation and control groups ($\chi^2(3) = 14.48$, $p < .01$). The number of D/I problems in the control group was significantly high but significantly low in the evaluation group was, whereas the number of I/D problems in the evaluation group was significantly high but significantly low in the control group. There was no difference between the reproduction-f and control groups ($\chi^2(3) = 4.64$, *n.s.*).

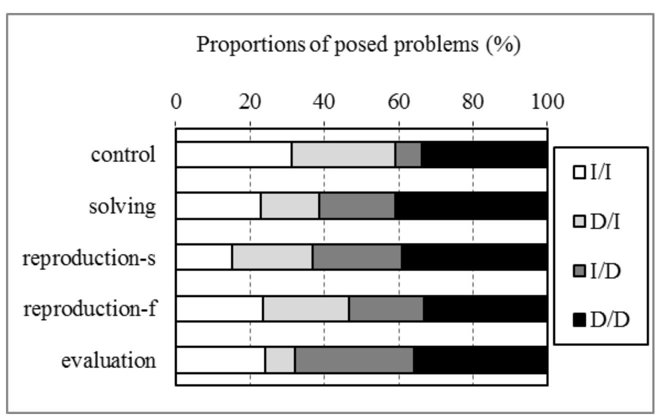

Figure 2: Proportions of posed problems in each category.

Solution-altering strategies

Figure 3 indicates the proportions of posed problems composed with each solution-altering strategy in each group. The chi-square test indicated a significant difference between the solving and control groups ($\chi^2(2) = 7.98$, $p < .05$). Furthermore, the results of residual analysis indicated that the number of not altered in the control group was significantly high but significantly low in the solving group, whereas the number of fully altered in the solving group was significantly high but significantly low in the control group. Similarly, there was a significant difference between the reproduction-s and control groups ($\chi^2(2) = 13.20$, $p < .01$). The results of residual analysis indicated that the number of not altered in the control group was significantly high but significantly low in the reproduction-s group. The number of partially altered in the reproduction-s group was significantly high but significantly low in the control group. There was also a significant difference between the evaluation and control groups ($\chi^2(2) = 8.20$, $p < .05$). The results of residual analysis indicated that the number of not altered in the control group was significantly high but significantly low in the evaluation group, whereas the number of partially altered in the evaluation group was significantly high but significantly low in the control group. There was a moderate but significant difference between the reproduction-f and control groups ($\chi^2(2) = 5.61$, $p < .10$). The results of residual analysis indicated that the number of partially altered in the reproduction-f group was significantly high but significantly low in the control group.

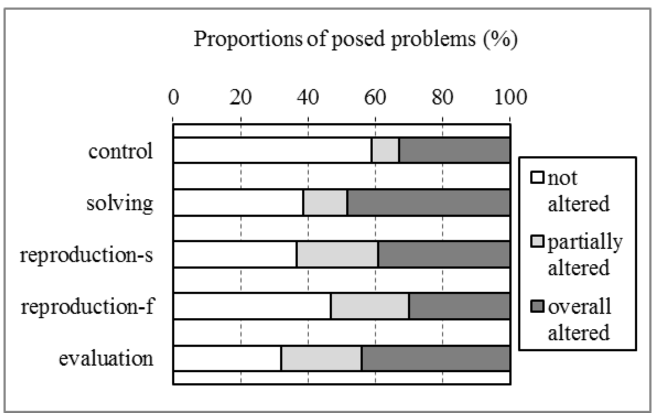

Figure 3: Proportions of posed problems with each solution-altering strategy

The complexities

Figure 4 indicates the proportions of I/D and D/D problems whose number of operations increased or decreased from the base. In half of the I/D and D/D problems posed by the control group, the number of operations decreased from the base. This means that half of the I/D and D/D problems were simpler than the base. The number of such simple problems was smaller only in the reproduction-s group. We compared the control group with the solving, reproduction-s, reproduction-f, or evaluation groups using the chi-square test; the results indicated a significant difference between the reproduction-s and control groups ($\chi^2(2) = 11.36$, $p < .01$). Furthermore, the results of residual analysis indicated that the number of decreased in the control group was significantly high but significantly low in the reproduction-s group. The number of increased in the reproduction-s group was significantly high but significantly low in the control group. There was no difference between the solving and control groups ($\chi^2(2) = 2.58$, *n.s.*), the reproduction-f and control groups ($\chi^2(2) = 1.06$, *n.s.*), or the evaluation and control groups ($\chi^2(2) = 0.06$, *n.s.*).

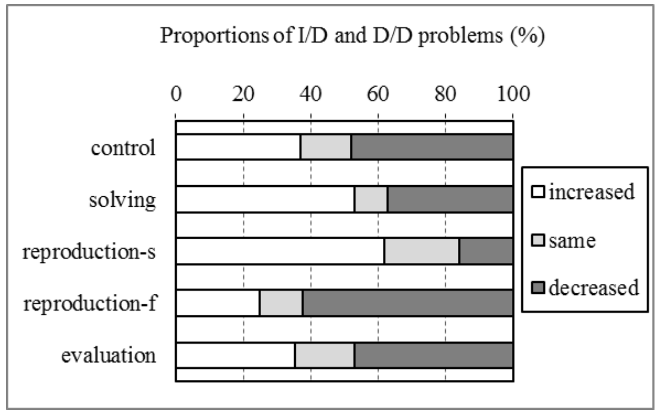

Figure 4: Proportions of altered problems whose operations increased or decreased

Discussion

The results presented above indicate that the experimental groups posed more I/D problems than the control group. The experimental groups learned the example of an I/D problem. Thus, the example facilitated posing problems in the same category of the example regardless of the learning activities.

On the other hand, there was a difference among the experimental groups in the solution-altering strategies. Problems posed with overall altered increased in the solving group, whereas problems posed with partially altered increased in the production-s, production-f, and evaluation groups. The latter three groups adapted ideas used in the example because it was composed with partially altered. The solving group learned the example through a comprehension task, while the reproduction-s and reproduction-f groups did so through a production task. It has been documented that the reproduction of an example can facilitate creative performance because such activity prompts a conceptual background in creation of the example (Ishibashi & Okada, 2006). Similar to reproduction, evaluation is regarded as a production task and related to creative thinking. In fact, the influence of evaluating examples on creative performance has been demonstrated (e.g., Lonergan, Scott & Mumford, 2004). Similarly, this study demonstrated that learning the example through a production task facilitated adaptation of the example to problem posing by the undergraduates.

The results shown in Figures 2 and 3 confirm that learning the example increased problems whose solutions were different from the base. As described in the introduction, it is difficult for novices to compose novel solutions in problem posing. The experimental groups posed problems with novel solutions in some senses, even though only the reproduction-s group posed many problems more complex than the base. The undergraduates could learn how to formulate more complex solutions by adding operations. However, such problem posing was performed only by those who had succeeded in reproducing the example.

These facts prove that learning by reproducing an example is effective in terms of a learning activity in a production task. However, this activity also involves difficulty. The reproduction-f group failed in the learning task. The example must be quite easy for undergraduates to solve. In fact, no one in the solving group failed in the learning task. Although learning by reproduction is effective, it imposes a significant challenge to learners. Therefore, further supportive intervention must be introduced in learning from an example through a production task.

The reproduction-s and the evaluation groups both adapted the example to the problem posing task. The evaluation group posed many I/D problems as well as partially altered problems. However, like the control group, the evaluation group posed many I/D and D/D problems that were simpler than the base. Although this group assessed the example from the viewpoints of its originality and feasibility, alternative viewpoints might be needed in evaluation to improve the effects of an example. Furthermore, to enhance the effects of evaluation, one alternative is to present a *nasty* problem as an example. A learner may devise a good idea through evaluating such an example and find how to improve it. Further study is needed to thoroughly examine this point.

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Appendix A: Information indicating how to compose A_2 from A_1

The example was composed by altering the base in the ways described below. According to these, make a problem identical to the example. It is unnecessary to exactly reproduce the text of the example as long as your problem is solved with the same solution.

x* & *y

objects are altered to "pencils" and "pens"

x: pencils

y: pens

answers: $x = 10$, $y = 3$ (how many)

Numeric parameters in text

2 parameters are added

parameters: (total)13, pen 110 yen, pencil 40 yen, pencil 2 times, (total) 1430 yen, pen case 300 yen

Third object (pen case and 300 yen) is added

Solution

Altered from the base

[*x* pencils] + [*y* pens] = [total 13]

[*1] × [*x* pencils] + [110 yen pen] × [*y* pens] = [*2]

*1 Operation [40 yen pencil] × [2 times pencils] is added

*2 Operation [total 1430 yen] − [pen case 300 yen] is added

Problem text

Keywords: last year, pencils, pens, total, buy, this year, the number, pen case

Appendix B: Example of posed problems

D/I problem posed in the solving group

A teacher planned to divide students into groups of equal numbers of students. If 5 students were assigned to each group, then 2 students were left. If 6 students were assigned to each group, then 4 additional students were needed. How many groups did the teacher want to make?

Solution.

Let *x* denote the number of groups.

$$5x + 2 = 6x - 4$$

$x = 6$.

I/D problems posed in the production-s group

To buy 8 loaves of breads, I need 100 yen more. If 30 percent is discounted from the price of a loaf, 284 yen is left after buying 8 loaves. Find the price of a loaf.

Solution.

$$8x - 100 = 8x \times (10 - 3) / 10 + 284$$

$x = 160$.

(posed with partially altered)

A store sells a "tasty cookie." A customer can buy a single cookie, and a bag containing some cookies. A family of 3 persons bought 6 bags and each person ate the same number of cookies. Another family of 6 persons bought 10 bags and 10 single cookies and each person ate the same number of cookies. The numbers of cookies for one person were the same in the both families. How many cookies does the bag contain?

Solution.

Let *x* denote the number of cookies in a bag.

$$6x / 3 = (10x + 10) / 6$$

$x = 5$.

(posed with overall altered)

D/D problems posed in the evaluation group

I drove from Tokyo to Nagoya. My car was driven at the speed of 100 km per hour on a highway and the journey took 4 hours. Find the distance I drove.

Let *x* denote the distance I drove.

$$100 \times 4 = x$$

$x = 400$.

(posed with overall altered)