# Calculus Expertise and Strategy Use when Comparing Multiple Representations

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**Abstract**

Expertise affords individuals a variety of advantages for learning and for problem solving, including competing advantages such as using automatic strategies vs. using sophisticated strategies. In the present study, high school students with varying levels of calculus expertise completed measures of conceptual understanding and skill with external representations before a task in which they were asked to coordinate between multiple representations (CMR) and determine whether they represented the same mathematical function. Strategy use during the CMR task was coded based on think-aloud data. Results indicate that students with more expertise tended to use automatic strategies when completing the task, and, surprisingly, used fewer sophisticated strategies than more novice peers.

**Keywords:** expertise; mathematics; coordinating multiple representations

## Introduction

Numerous studies have documented the advantages that experts in a domain have for performance and learning in that domain. These advantages include, but are not limited to, improved memory abilities (Chi, 1978), knowledge of strategies (Gaultney, 1995), and reasoning skills (Johnson, Scott, & Mervis, 2004). Further, such benefits have been shown to arise in widely varied domains of expertise, including physics (Chi, Feltovich, & Glaser, 1981), dinosaurs (Johnson, Scott, & Mervis, 2004), chess (Chi, 1978), computer programming (Barfield, 1986), and gymnastics (Tenenbaum, Tehan, Stewart, & Christensen, 1999).

Regardless of domain, several of these advantages have direct implications for experts' problem solving skills. Perhaps the most obvious implications follow from improved strategy knowledge. By definition, experts have a greater conceptual understanding within their domain, and they are able to use this knowledge to generate better strategies for problem-solving in that domain (Chi et al., 1981). Their strategies tend to be driven by theory and consistent with their conceptual understanding of the problem situation (Dhillon, 1998). In contrast, novices tend to think about problems in a more basic manner, with strategies that focus on surface structures rather than on deep ones (Lovett & Anderson, 1996). Overlapping waves theory maintains that all individuals know and use a variety of strategies for solving problems, and that those strategies compete for use in every given situation; as individuals become more experienced and gain expertise, they are likely to use more sophisticated strategies than peers with less expertise (Siegler, 1996). Experts' greater success in problem solving can be, at least partially, attributed to their use of more sophisticated strategies (Chi, Glaser, & Rees, 1982).

Another key advantage for experts that is relevant to problem solving is experts' automaticity—that they have automatized certain knowledge and thus do not have to spend limited working memory resources to process problem components or carry out the procedures (Bereiter & Scardamalia, 1993). Such procedures are thus carried out without the individual consciously monitoring them (Aarts & Dijksterhuis, 2003), allowing experts to perform better on difficult problem-solving tasks (Brown & Bennett, 2002). Automaticity can also happen in interpretation of the problem situation (Bargh, 1999), which is arguably related to another advantage of expertise: rapid and effective encoding of relevant information (Ericsson & Kintsch, 2000). Both automaticity and encoding are core mechanisms in information processing theories, indicating ways in which more experienced individuals are able to bypass capacity limitations in order to focus on critical problem features (e.g., LaBerge & Samuels, 1974; Heatherington & Parke, 2003). These mechanisms may also help account for the fact that experts often have difficulty articulating the strategies they utilize when solving problems (Chao & Salvendy, 1994).

In the present study, we investigate how mathematics expertise relates to students' approaches to the problem of coordinating multiple representations (CMR) in mathematics. The National Council for Teachers of Mathematics recommends use of external representations throughout mathematics instruction, even in elementary school, and stresses the importance of being able to coordinate between multiple representations (NCTM, 2010). This skill is especially emphasized in reform approaches to teaching higher-level mathematics courses, such as calculus (e.g. Hughes-Hallett et al., 2011). Here, we examine specifically how conceptual knowledge in calculus (expertise) relates to students' use of automatic vs. sophisticated vs. basic strategies for completing a CMR task. We also test the degree to which conceptual knowledge is predictive of strategy use above and beyond other factors which could increase performance (grade level and skill with external representations). Based on the literature described above, we expect that students with greater conceptual knowledge will use sophisticated strategies (Siegler, 1996) or automatic strategies (Bereiter & Scardamalia, 1993), but not basic ones. However, it is unknown whether they are more likely to use automatic or sophisticated strategies when coordinating multiple representations.

# Method

## Participants

Participants included 40 pre-calculus and calculus students from two public, suburban high schools in the Northeast. Their mean age was 16.6; 45% were male; 77% were White, 18% Asian, 3% Black, and 5% other races. As a proxy for socioeconomic status, median parental education was a Bachelor's degree.

## Procedure

Parental consent and student assent were acquired, after which students were tested individually in a session lasting approximately 70 minutes. Students received gift cards as compensation in the amount of $10 for their involvement in the study. Participants completed a series of paper and pencil measures, including a standardized measure of basic graph/table skills and one researcher-constructed calculus conceptual knowledge measure. Students also completed an eye-tracking measure for coordination of multiple representations (CMR) while verbalizing their thoughts. These measures are described below.

## Measures

**Calculus conceptual measure (CCM).** To assess students' calculus conceptual knowledge, we used 32 researcher-constructed items that measured students' understanding of concepts that have been identified as crucial for success in calculus, including functions and limits (Lauten, Graham, & Ferrini-Mundy,1994), derivatives (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997), and the chain rule (Clark et al., 1997). For example, one item asks participants to identify which pieces of information they would use to complete a mathematical task (e.g., finding the zeros of a function); students respond by circling any number of the following options; *f, f', and f''*. Students were given 7 minutes to complete the measure. Scores ranged from 0% to 72% (M=47%) and were highly correlated with a measure of students' proficiency in calculus (Cromley et al., revision under review). Internal consistency was computed using Cronbach's alpha; $\alpha$= .99.

**Graph/table skills.** We created a measure comprised of 6 released NAEP Grade 12 graph items and 5 released NAEP, NAAL, and ALL table items. These are from the "Easy" groups of multiple-choice items, which tap basic graph and table comprehension, such as finding a single data point/cell, rather than coordinating multiple representations skills which are typical of many "Hard" and some "Medium" questions. One item asks participants to find the highest temperature in a city from a graph of temperatures across a range of days. Given these are well-validated questions used by NAEP for years before public release, we expected them to show excellent reliability and validity with high school students and undergraduates. Participants were given 6 minutes to complete the measure. Cronbach's alpha for the sample is .66.

**Think-aloud CMR Measure.** Student participants completed CMR tasks on an eye-tracking apparatus (eye-tracking data reported elsewhere; Wills, Shipley, Chang, Cromley, & Booth, in press) while following a think-aloud protocol (Ericsson & Simon, 1998). Participants were shown 12 pairs of representations and were asked to determine whether or not they expressed the same function. We used linear, quadratic, and cubic functions, with an equal number of matches and mismatches and equal numbers of the three possible pairings of each representation type (i.e., equation-graph, equation-table, graph-table). Position of the first representation on the left or right was controlled for across participants. A sample item is shown in Figure 1. Cronbach's alpha for responses to the match/mismatch questions for the sample is .87.

## Coding of Think-aloud Data

Student verbalizations during the CMR task were transcribed and analyzed to develop a coding scheme for

Table 1: Think-aloud codes for automatic, sophisticated, and basic strategy use during the CMR task.

| Strategy Code | Description | Example |
|---|---|---|
| Automatic | | |
| AHA | Feeling of knowing or "Aha!" moment | *oh, alright, f(x) equals y* |
| IDK | Reflecting on not knowing how to work through a stimulus | *I'm not really sure how to determine if they are the same because I haven't seen a graph like that before* |
| Sophisticated | | |
| EVALDIR | Evaluating the direction (positive or negative) | *I see that this parabola is facing downwards so I would say that that's negative and it's not negative in the function* |
| EVALCOEFF | Evaluating the magnitude of a coefficent | *The slope is 2 in the equation, rise 2 over 1, looks good in the graph* |
| EVALCON | Evaluating the magnitude of the constant | *It looks like this graph is shifted to the right* |
| EVALOR | Evaluating the order | *To me that doesn't look like an x squared function, that looks to be like x cubed.* |
| Basic | | |
| MOP | Mapping an ordered pair between representations | *Let's see, x is 2 on the table, put it on the graph, equals negative 20, that fits with the table* |
| MX | Mapping the x intercept between representations (with or without verbally identifying it as such) | *x equals 3 and y equals 0, that matches the graph* |
| MY | Mapping the y intercept between representations (with or without verbally identifying it as such) | *So when x is zero, f of x is around negative one* |

student strategy use. The second author coded 1,370 utterances across all participants (*M* = 34 utterances per participant). A second coder was trained on data other than those used for calculating inter-rater statistics and re-coded 35% of the corpus (Cohen's kappa = .91). See Table 1 for a description of the think-aloud codes used in the present analyses. Two of the codes (AHA and IDK) represent automatic strategies, in which overt procedures are not undertaken to solve the problem. Four of the codes (EVALDIR, EVALCOEFF, EVALCON, and EVALOR represent sophisticated strategies which invoke deep problem features (the direction, coefficient, constant, or order of the function) in the solution process. The remaining three codes (MOP, MX, MY) reflect more basic problem solving strategies, in which only surface features in the problem (namely, the discrete ordered pairs in the functions) are utilized.

## Results

First, to determine which types of strategies should be further explored with relation to conceptual understanding, we computed the mean number of times per item that verbal data indicated the use of each strategy. We then computed correlations between the CCM scores and the mean use of each strategy. As shown in Table 2, significant correlations were found between CCM scores and the AHA, IDK, EVALCOEFF, and EVALCON strategies. Thus, all further analyses will be conducted only on these strategies.

To determine whether CCM is a useful and independent predictor of the use of each of these strategies, and to further investigate the nature of its impact, we conducted a series of regression analyses, one for each of the strategies with

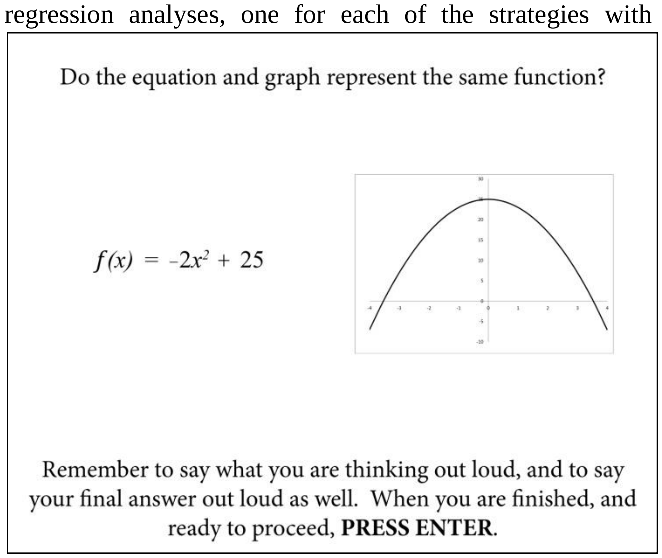


Figure 1: A sample item from the eye tracking CMR task. This example pairs an equation with a graph; other items included equation-table or graph-table pairs.

Table 2: Correlations between conceptual understanding and automatic, sophisticated, and basic strategy use.

| | Automatic | | Sophisticated | | | | Basic | | |
|---|---|---|---|---|---|---|---|---|---|
| | AHA | IDK | EVALDIR | EVALCOEFF | EVALCON | EVALOR | MOP | MX | MY |
| CCM | .36** | -.38** | .18 | -.47*** | -.35** | .24 | -.02 | .05 | .14 |

Note: ****p*<.01, ***p*<.05

Table 3: Regression results for conceptual understanding (CCM), grade level, and graph/table skills (NAEP) on use of each strategy

| | CCM | | Grade | | NAEP | |
|---|---|---|---|---|---|---|
| Analysis | β | Significance | β | Significance | β | Significance |
| AHA | .36 | $p < .05$ | .04 | *ns* | -.12 | *ns* |
| IDK | -.43 | $p<.01$ | .26 | $p<.10$ | -.25 | $p<.10$ |
| EVALCOEFF | -.49 | $p<.01$ | .00 | *ns* | .11 | *ns* |
| EVALCON | -.31 | $p<.10$ | -.15 | *ns* | .11 | *ns* |

which CCM scores were correlated. In each model, we also included two alternate predictors: grade level and NAEP scores. Grade level was included because students in higher grade levels tend to have higher conceptual knowledge ($r(38) = .33$, $p < .05$), and being in an advanced grade may also lead to more sophisticated strategy use, as more sophisticated strategies are likely to be taught in higher grades. NAEP graph/table scores were included in order to isolate the impact of domain-specific expertise (e.g., conceptual knowledge in calculus) rather than more general competence with external representations (e.g., graphs and tables) which may also come with increased experience in mathematics classes.

As shown in Table 3, conceptual knowledge in calculus was a significant positive predictor of the AHA strategy and the MOPY strategy, whereas it was a negative predictor of the IDK strategy and the EVALCOEFF strategy, and a marginal negative predictor of the EVALCON strategy. Students with higher conceptual knowledge of calculus were thus more likely to have an "Aha" moment or to match ordered pairs for the y-intercept, and less likely to evaluate the magnitude of the coefficient or the constant, or to say they did not know. Grade and NAEP graph/table scores proved to be marginal predictors of the IDK strategy (e.g., students with higher NAEP scores were less likely to say they don't know, but students in higher grades are more likely to say they don't know), but in all cases, there was an impact of conceptual understanding above and beyond that of the other factors.

## Discussion

Results from this study suggest that students with greater conceptual understanding do not, in fact, use more sophisticated overt strategies than peers with less conceptual knowledge. Instead, our results indicate that more conceptually strong students tend to complete the task without using overt strategies at all. These students are most likely to just have an "Aha!" moment in which they feel that they know whether the functions are the same.

Conceptually strong students, however, do not just use any automatic strategy. They are less likely than more novice peers to utilize a relatively unproductive automatic strategy—giving up and saying they do not know whether the functions are the same.

While it may not be surprising that experts in this study preferred productive automatic to overt sophisticated strategies, it is perhaps perplexing that the relation between expertise and use of overt sophisticated strategies was, in fact, negative. Experts used sophisticated strategies *less often* than more novice peers. This seems contrary to the wealth of literature on experts' better repertoire of strategies. However, it is important to note that these results do not speak to whether or not more advanced students *know* more sophisticated strategies, just whether or not they seem to *use* them overtly during the comparison tasks. The underlying procedures that they are using to automatically process and solve the problem may or may not be sophisticated; it is impossible to tell that from the think aloud data. Further analysis of eye-tracking data collected during the CMR task will be necessary for distinguishing which problem feature(s) the conceptually strong students noticed before and while they experienced the feeling that they knew the answer. Analysis of the eye-tracking data may also allow us to determine whether experts' decisions were made based on automatic processing of problem features (encoding differences) or automatic application of problem-solving procedures.

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## References

Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. E. (1997). The development of students' graphical understanding of the derivative. *Journal of Mathematical Behavior, 16*, 399–431.

Barfield, W. (1986). Expert-novice differences for software: Implications for problem solving and knowledge acquisition. *Behaviour and Information Technology, 5,* 15-29.

Bargh, J. A. (1999a). The unbearable automaticity of being. American Psychologist, 54(7), 462–479.

Bereiter, C., & Scardamalia, M. (1993). *Surpassing ourselves: An inquiry into the nature and implications of expertise*. Chicago, IL: Open Court.

Brown, S. W., & Bennett, E. D. (2002). The role of practice and automaticity in temporal and nontemporal dual-task performance. *Psychological Research*, *66*, 80–89.

Chao, C., & Salvendy, G. (1994). Percentage of procedural knowledge acquired as a function of the number of experts from whom knowledge is acquired for diagnosis, debugging, and interpretation tasks. *International Journal of Human-Computer Interaction, 6*(3), 221–233.

Chi, M.T.H. (1978). Knowledge structures and memory development. In R.S. Siegler (Ed.), *Children's thinking: What develops?* (pp. 73-96). Hillsdale, NJ: Erlbaum.

Chi, M. T. H., Feltovich, P. J., & Glaser, R. (1981). Categorization and representation of physics problems by experts and novices. *Cognitive Science*, *5*, 121–152.

Chi, M. T. H., Glaser, R., & Rees, E. (1982). Expertise in problem solving. In R. J. Sternberg (Ed.), *Advances in the psychology of human intelligence, vol. 1.* Hillsdale, NJ: Erlbaum.

Clark, J. M., Cordero, F., Cottrill, J., Czarnocha, B., DeVries, D. J., St. John, D., Tolias, T., & Vidakovi´c, D. (1997). Constructing a schema: The case of the chain rule. *Journal of Mathematical Behavior, 16*, 345–364.

Cromley, J.G., Booth, J.L., Wills, T.W., Chang, B.L., Shipley, T.F., Zahner, W., Tran, N., & Madeja, M. (revision under review). Spatial ability, conceptual knowledge, and calculus proficiency. *Learning and Instruction.*

Dhillon, A. S. (1998). Individual differences within problem-solving strategies used in physics. *Science Education*, *82,* 379–405.

Ericsson, K. A., & Kintsch, W. (2000). Shortcomings of generic retrieval structures with slots of the type that Gobet (1993) proposed and modeled. *British Journal of Psychology*, *91*, 571–590.

Ericsson, K., & Simon, H. (1980). Verbal reports as data. *Psychological Review 87* (3): 215–251.

Gaultney, J.F. (1995). The effect of prior knowledge and metacognition on the acquisition of a reading comprehension strategy. *Journal of Experimental Child Psychology*, *59*, 142-163.

Hetherington, E.M., & Parke, R.D. (2003). *Child Psychology: A Contemporary Viewpoint (Updated Fifth Edition)*. Boston: McGraw-Hill.

Hughes-Hallett, D., Gleason, A. M., Lock, P. F., & et al. (2010). *Applied calculus* (4th ed.). Hoboken, NJ: Wiley.

Johnson, K.E., Scott, P., & Mervis, C.B. (2004). What are theories for? Concept use throughout the continuum of dinosaur expertise. *Journal of Experimental Child Psychology*, *87*, 171-200.

Laberge, D., & Samuels, S. J. (1974*). Towards a theory of automatic information processing in reading. Cognitive Psychology*, *6*, 293-323.

Larkin, J. H., McDermott, J., Simon, D. P., & Simon, H. A. (1980). Expert and novice performance in solving physics problems. *Science*, *208*, 1335–1342.

Lauten, A. D., Graham, K., & Ferrini-Mundy, J. (1994). Student understanding of basic calculus concepts: Interaction with the graphics calculator. *Journal of Mathematical Behavior*, *13*, 225–237.

Lovett, M. C., & Anderson, J. R. (1996). History of success and current context in problem solving: Combined influences on operator selection. *Cognitive Psychology*, *31*, 168–217.

National Council of Teachers of Mathematics. (2010). *Principles and standards for school mathematics*. Reston, VA: Author.

Siegler, R. S. (1996). *Emerging minds: The process of change in children's thinking.* New York: Oxford University Press.

Tenenbaum, G., Tehan, G., Stewart, G., & Christensen, S. (1999). Recalling a floor routine: The effects of skill and age on memory for order. *Applied Cognitive Psychology, 13*, 101-123.

Wills, T., Shipley, T. F., Chang, B. L., Cromley, J. G., & Booth, J. L. (in press). What gaze data reveal about coordinating multiple mathematical representations. Proceedings of the 36th Annual Meeting of the Cognitive Science Society.