

Equations are Effects: Using Causal Contrasts to Support Algebra Learning

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Abstract

U.S. students consistently score poorly on international mathematics assessments. One reason is their tendency to approach mathematics learning by memorizing steps in a solution procedure, without understanding the purpose of each step. As a result, students are often unable to flexibly transfer their knowledge to novel problems. Whereas mathematics is traditionally taught using explicit instruction to convey analytic knowledge, here we propose the *causal contrast* approach, an instructional method that recruits an implicit empirical-learning process to help students discover the reasons underlying mathematical procedures. For a topic in high-school algebra, we tested the causal contrast approach against an enhanced traditional approach, controlling for conceptual information conveyed, feedback, and practice. The causal contrast approach yielded remarkably greater success, especially on novel problems, across students with varying levels of mathematical competence.

Keywords: comparison; mathematics education; causal induction; concepts, knowledge transfer

Introduction

Mathematics experts use general goal-directed reasoning to solve problems. Students, in contrast, are commonly observed to use procedures without understanding, often just following a memorized sequence of steps to solve a problem (Stigler, Givvin, & Thompson, 2010). Because they have not connected procedures to goals or concepts, students lack the flexibility of experts: they often are unable to decompose a procedure and apply the constituent operations successfully to a novel problem for which the learned sequence is inadequate (Cooper & Sweller, 1987). This lack of goal-directed reasoning in mathematics may be a major cause of the poor performance of U.S. students (PISA, 2009, 2013; Stigler & Hiebert, 1999; TIMSS, 2007).

A critical prerequisite for goal-directed reasoning is learning cause-and-effect relations. People, including children and even infants, have a natural capacity for learning causal relations (Cheng, 1997; Gopnik et al, 2004; Leslie & Keeble, 1987). Much like language learning, causal induction is a universal learning process that is shared by virtually all humans. Because of its universality, causal induction can be recruited in nearly *all* learners to support goal-directed reasoning. Once an effect of interest is identified, most learners will naturally seek to discover causes of that effect. Moreover, they will succeed if the requisite information for the discovery is readily available. For example, imagine you wake up one morning,

disappointed to discover that your digital video recorder (DVR) failed to record a special television show last night (your goal). You would probably think back to occasions on which your DVR successfully recorded, and compare them to the failed attempt. If a presetting to record a regular show is the only feature that differs between your failed and successful attempts, you would readily determine the cause of the failure -- the presetting interfered with your new setting. This process enables you to discover a new causal relation and better understand how your DVR works.

Causal induction is the process whereby we come to know how the empirical world works; it is what allows us to predict, diagnose, and intervene on the world to achieve an effect. Based on the premise that action-outcome relations in mathematics are no less causal than those in other domains, we ask: Can we 1) activate students' natural capacity for causal induction during mathematical problem-solving, inducing them to formulate the goals of a mathematical procedure, and 2) promptly have the requisite information available, allowing students to discover the actions that would achieve those goals, and thereby to understand the causal structure of the solution to a problem? If so, students may become better able to generalize their mathematical knowledge. Although previous researchers have noted the importance of applying causal knowledge to solve mathematical problems (e.g., Anderson, 1990), to our knowledge the simple but powerful process of causal induction has never been systematically recruited in the acquisition of the requisite causal knowledge in mathematics education. Causal induction may at first appear irrelevant to mathematical learning, as mathematics is analytic whereas causation is empirical. Moreover, the causal induction process is implicit; neither its requisite input nor its operation is open to introspection. Thus, the potential use of this process in mathematics learning does not readily present itself.

Traditional instructional approaches teach students analytically and explicitly the rules and steps for solving specific types of problems. For example, a student presented with this equation

$$x^2 - 13x - 30 = 0 \quad (\text{Eq. 1})$$

is taught to first factor the expression on the left, then determine the possible values of x using the *zero-product property* (ZPP) (if $a \cdot b = 0$, then $a = 0$ or $b = 0$). For a substantial fraction of students, as evidenced by their failure to flexibly generalize their learning to novel problems, this

approach does not impart an understanding of the causal structure of the solution, the reason behind each step in the procedure and how the steps work together. For example, what is the purpose of factoring the expression on the left hand side (LHS) -- why can't one merely rearrange the equation to solve for x ? And what is the relationship between factoring and the ZPP? Even when students are given opportunities to compare worked examples (Rittle-Johnson & Star, 2007; Rittle-Johnson, Star & Durkin, 2009), or explain solutions (Chi et al, 1989, 1994; Chi, 2000), they may still fail to formulate questions that enable understanding of the causal structure of a problem.

The approach we are exploring is designed to support students' discovery of the causal structure that underlies mathematical procedures, the causal structure that reflects the critical mathematical concepts essential to the solution of relevant types of problems. At the heart of our approach is the use of targeted comparison tasks designed to activate students' cause-effect learning at just the points in the problem-solving process at which critical concepts should apply. It is not the use of comparison *per se* that differentiates our approach, but rather the targeting of critical concepts by specific comparisons designed to invoke causal induction. To continue with the same example, students in our approach are first asked to try solving Equation 1. If they fail to solve Equation 1, we present them with a *contrasting problem*, one that is carefully chosen to control for confounding factors by being as similar as possible to Equation 1 but with the features causing the difficulty removed. The contrast aims at invoking students' implicit causal learning process, so that they discover the cause of their difficulty. For example, after failing to solve Eq. 1, students are asked to solve these equations:

$$x - 30 = 0 \quad (\text{Eq. 2})$$

$$x^2 - 30 = 0 \quad (\text{Eq. 3})$$

After students solve these (all our participants did), they are asked why these problems are easier for them to solve than Equation 1. This comparison enables students to readily discover a cause of their failure: unlike Equations 2 and 3, Equation 1 has both an x and an x^2 term, preventing them from isolating x by simply rearranging the equation. The newly formulated cause -- having both x and x^2 terms -- in turn becomes an "effect" for subsequent operations to prevent or remove. Additional contrast comparisons enable students to discover a conjunctive preventive relation, namely, that factoring and applying the ZPP in combination can result in equations that can be rearranged to isolate x .

Figure 1 displays a fragment of the causal structure of the solution for Equation 1. Whereas causal arrows point from the bottom up in the figure, learners construct the causal structure by experiencing the contrasting events from the top down. Leaving out any "cause" or "effect" would create gaps in one's mental causal structure of the solution. Without their initial attempt to isolate x , learners would have no "effect" (top node in the figure) for which to discover its cause. This effect -- failure to isolate x by rearranging the equation -- is not explicitly noted in

traditional instruction on solving quadratic equations and requires formulation. Conversely, without a comparison with Equations 2 and 3, a learner who is asked to explain why Equation 1 is difficult may mention the effect alone, "I'm unable to rearrange the equation to isolate x ", omitting to identify its "cause": having both x and x^2 terms in the same equation, the intermediate node. In turn, without the intermediate node, there would be no "effect" for learners to subsequently discover its preventive cause (the bottom node), namely, factoring. Thus, each comparison is designed to direct attention to an essential causal relation in the structure of the solution.

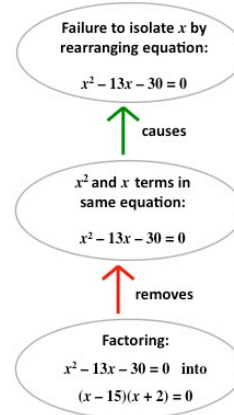


Figure 1. Fragment of the causal structure of solution to a factorable quadratic equation.

In more general terms, the comparisons we construct provide *conditional contingency information* (Cheng & Holyoak, 1995), input that enables a *causal contrast*. For events involving causes and effects that are binary -- the type of events of concern here -- conditional contingency information consists of the state of an effect (e.g., succeeding to solve a problem or not) in the presence of a candidate cause (a feature of the math problem, e.g., an equation having both an x and an x^2 term) and in its absence (an equation *not* having both an x and an x^2 term), with alternative causes held constant.

The process repeats as the student proceeds through the solution. Each step involves 1) the student's attempt to solve a difficult problem that presents an opportunity for failure -- a misunderstanding of a critical concept or an impasse -- and 2) a comparison of the difficult problem with a contrasting easier problem. The failures identify the effects and the comparisons provide conditional-contingency information necessary for discovering their causes.

Continuing with our example problem, suppose that when we ask students to solve a factored version of Equation 1,

$$(x - 15)(x + 2) = 0, \quad (\text{Eq. 4})$$

they do not encounter an impasse. They proceed to make use of the ZPP and infer that

$$x - 15 = 0 \quad \text{or} \quad (\text{Eq. 5})$$

$$x + 2 = 0. \quad (\text{Eq. 6})$$

For some students, this success may indicate complete understanding of the causal structure. But for many students it may not. For example, these same students, if given the equation

$$(x - 2)(x + 2) = 12, \quad (\text{Eq. 7})$$

might erroneously infer that

$$(x - 2) = 12 \text{ and} \quad (\text{Eq. 8})$$

$$(x + 2) = 12. \quad (\text{Eq. 9})$$

In fact, this represents a prevalent misconception; roughly half of our subjects (72% community-college students and 39% university students) committed this error on the pretest.

To address this misconception, we ask students to compare Equation 4 with the easier contrasting problem

$$x \cdot y = 0. \quad (\text{Eq. 10})$$

We ask, “what values of x would make $x \cdot y = 0$ true, regardless of the value of y ?” In this simple form, students are able to think through the logic of the ZPP; all our participants answered the question correctly. We then ask them to compare Equation 10 with Equation 4. Solving Equation 10 and comparing it to Equation 4 enables these students to see the analogous structure underlying the two equations and discover their misconception of the ZPP. Thus, what is critical to learning from causal contrasts is a misconception of the causal structure of a difficult problem, whether that misconception is manifested in an impasse (as in Equation 1) or succeeding for the wrong reason (as in Equations 7, 8, and 9).

For students who are at an impasse, unable to solve Equation 4, the comparison with Equation 10 may allow them to discover why they failed to transfer, for example, that the perceptual complexity of Equation 4 prevented them from recognizing that the equation is a product, that the content of the parentheses is “just a number”. At this point, the perceptual complexity can switch causal roles, taking on the role of an effect that the student can prevent or remove. The students may in the future pause to “see through” the perceptual complexity to recognize when a concept applies.

To summarize, the particular sequence of problems and contrast comparisons in our materials were designed to reveal the causal structure of the solution by focusing students on the goals of procedures at various levels of abstraction: why they factor, why they apply the ZPP, and perceptual cues indicating when the property applies. Our approach enhances the causal structure of the solution procedure whenever it is incomplete. Understanding the causal structure rather than memorizing the procedure enables generalization and transfer.

Causal Contrast Compared to Similar Methods

Although our approach builds on previous methods in the literature (Anderson, 1990; Bjork, 1994; Chi et al, 1989; Chi et al, 1994; Chi, 2000; Kornell, Hays, & Bjork, 2009; VanLehn, 1988), no previous study has tested learning via causal-contrast comparisons. Unlike our materials, materials in previous studies were not designed to provide conditional-contingency information for the steps in a solution procedure.

Rittle-Johnson and Star (2007) had students compare and contrast alternative solution procedures to a worked example (algebra problems), or two different worked examples that use the same solution procedure (Rittle-Johnson et al, 2009). Although the comparison of alternative solutions leads to an understanding of the common and distinct features of alternative solutions, it is not targeted at enhancing the learning of the purposes underlying individual steps. Without the requisite conditional-contingency information for the steps, students may still fail to transfer to novel problems. Indeed, there is no evidence that solution-procedure comparisons induced transfer; the posttest problems in Rittle-Johnson and Star (2007) differed only slightly from the worked examples appearing in the instruction, and could be solved using the same procedures. Notably, what they term “conceptual problems” were solved equally well by the comparison and no-comparison groups. Thus, while our approach shares with Rittle-Johnson et al. the use of comparison tasks to promote learning, the two approaches differ both in what is compared and which type of reasoning process they engage.

Anderson (1990) argues that causal knowledge is used to select the operators to achieve each step in a solution. Unlike our approach, however, his does not give learners the contingency information necessary to discover the reasons for the steps. Instead, learners are simply encouraged to *retrieve* the causal links from memory and apply them onto the current situation. Indeed, Anderson (1987) admits that his approach does not afford students the ability to infer the reasons for the procedure.

The use of conditional-contingency information also distinguishes our approach from previous work on the role of self-explanation in learning. There is a large body of work that shows that simply giving a prompt of “please explain” or “why?” during learning can result in an enhanced ability to generalize (Chi et al, 1989; Chi et al, 1994). This *self-explanation effect* is especially pronounced when learners experience an impasse prior to their explanation (Chi, 2000; VanLehn, 1988). Chi (2000) has argued that this effect occurs because self-explaining helps learners find and correct inaccuracies in their mental models of the particular domain under study. However, as we illustrated earlier regarding a self explanation of Equation 1 without a comparison with Equations 2 and 3, self explanations formed in the absence of conditional-contingency information may be too general and do not ensure closing gaps in the causal structure.

The enhancement of the self-explanation effect by impasses is part of a large literature showing more generally that allowing learners to reach an impasse during problem-solving enhances transfer to novel problems (VanLehn, 1988). Related work in the memory literature indicates that encountering difficulties (e.g., retrieving wrong answers) during study improves performance at test (Bjork, 1994; Kornell, Hays, & Bjork, 2009). While impasses and difficulties may enhance learning, they are not necessary for learners to benefit from causal contrasts. What drives

learning in the causal-contrast approach is awareness of a gap in the causal structure of the solution to a problem; an impasse simply serves as a means to alert learners to the gap. As we illustrated with our example contrasts (the comparison between Eqs 4 and 7), causal contrasts can alert learners to the gap even when learners do *not* experience an impasse. Finally, impasses without the requisite conditional-contingency information are insufficient for ensuring the construction of a complete causal structure.

In summary, the present article illustrates a causal contrast approach to teaching mathematics. By recruiting a universal learning process through the use of conditional-contingency information in the instructional materials to elicit causal-contrast comparisons, the approach enables students to formulate a causal structure of the solution. We hypothesize that discovering the causal structure underlying a solution procedure – a) the goals and their interrelationships and b) the operations to achieve each goal – should enhance even struggling students’ ability to generalize and use the procedure more flexibly. We tested our approach on community-college students taking remedial algebra as well as on university students (to test the generality of our findings). Although our illustrative tests focus on algebra, the method we are testing is general and can be applied to other areas of math.

Comparing Causal-Contrast and Traditional Instruction

This experiment compared the effect of a causal-contrast approach to teaching algebra (as developed above) with a traditional approach, on learning and transfer. A between-subjects pretest/intervention/delayed-posttest design was employed, with the interventions focusing on solving factorable quadratic equations.

Method

Participants. Participants included (N=47) community college students recruited through College Algebra courses in Southern California, and university students (N=66) recruited from the UCLA Psychology Department undergraduate subject pool. Participants were randomly assigned to an intervention condition after the pretest¹: causal-contrast (n=50) and traditional (n=63).

Instructional Conditions. Students in the causal-contrast condition were given a packet with “difficult” problems – problems they were likely to fail – presented together with their associated contrasts. Difficult problems were selected to represent common misconceptions, and included $x^2 = 25$, $x^2 - 13x - 30 = 0$ (Eq. 1), and $3x^2 - 10x + 8 = 0$. Carefully

following a script, the experimenter directed students’ attention to the problems and comparisons, and gave feedback only to indicate whether the student solved a problem correctly. Each difficult problem-contrast set was followed by 2 or 3 practice problems that received no feedback.

The traditional intervention was designed to make use of techniques representative of excellent, but traditional, instruction. Students in the traditional condition were given a packet of instruction based on a popular textbook (Sullivan & Sullivan, 2007) and lesson plans from mathematics teachers at an academically rigorous private school. The instruction packet included *worked examples*, written solutions of example problems that provide justifications for each procedural step, followed by problem solving with and without feedback. Carefully following a script, the experimenter went step-by-step through the worked examples with each subject, stating the “sub-goals” of the critical steps in the procedure; for example, subjects were told that a quadratic equation is rearranged to standard form (having a zero on one side and a polynomial on the other) so that it could be factored and solved using the ZPP. Emphasizing sub-goals in problem-solving has been shown to enhance transfer (Eiriksdottir & Catrambone, 2011). When a student solved a problem incorrectly, the experimenter demonstrated how to solve the problem using the procedure in the worked examples. Training that combines worked examples and problem solving has been shown to enhance learning (Sweller & Cooper, 1985).

The interventions were identical except for the instructional method. Both conditions contained the exact same problems, in the same order, with the same subset of problems receiving feedback in both conditions. Both interventions lasted an average of 30 minutes.

Posttest. The delayed posttest occurred 2 to 3 weeks after the intervention and took at maximum 30 minutes. It included two types of problems: *instructed* and *transfer*. Instructed problems could be solved using the same solution procedures as the study problems. Transfer problems required generalization of concepts learned in the intervention. For example, one transfer problem,

$$x^2(2x + 1)(x + 1) = 0 \quad (\text{Eq. 11})$$

tests generalization of the role of the ZPP. In particular, the problem aims to test whether causal-contrast participants truly understood the difficulty in having both x and x^2 terms in a problem (a difficulty not present in Equation 11). Would they be stymied, for example, because the factored equation still contains x^2 ? Other transfer problems included factorable quadratics in non-standard form (see Figure 3).

If causal contrasts can result in an understanding of the underlying purposes of procedures, participants in the contrast group should outperform those in the traditional group, especially on the transfer problems; they should be able to use their knowledge flexibly to solve novel problems. The instructed problems allow us to check the effectiveness of our traditional intervention.

¹ Participants with ceiling or floor (below 20%) scores on the pretest were thanked for their time and not invited to participate in the study (13% of community college students excluded for ceiling scores, 7% for low scores; 15% of university students excluded for ceiling scores, 1% for low scores); the extremely low scores indicated difficulty with basic mathematical notation.

Results

Pretest. The causal-contrast and traditional groups did not differ in pretest performance for both the community-college students ($M_{contrast} = 56.1 \pm 15.2$; $M_{traditional} = 54.1 \pm 18.4$), $t(45) = .64$, $p = .45$, or university students ($M_{contrast} = 62.6 \pm 13.5$; $M_{traditional} = 64.3 \pm 16.7$), $t(64) = 1.32$, $p = .67$. Error terms indicate one S.D.

Posttest. For each sample, we conducted a one-way MANCOVA with two levels of instruction (causal-contrast, traditional) and two dependent measures (transfer and instructed problem performance), using pretest score as a covariate.

For the community college students, pretest score was marginally related to posttest performance, $F(1, 44) = 2.44$, $p = .10$. Figure 2 (left panel) displays the posttest performance data (adjusted means) for the community-college students. As the left panel of the figure indicates, the between-group difference on the transfer problems was remarkable ($M_{contrast} = 63.7 \pm 29.3$ and $M_{traditional} = 36.1 \pm 28.7$; $F(1, 44) = 9.63$, $p = .003$, $d = .95$). Whereas only about a third of the transfer problems were solved in the traditional group, almost two thirds were solved in the causal-contrast group. The instructed problems were also better solved by the contrast group ($M_{contrast} = 89.2 \pm 13.2$, $M_{traditional} = 70.4 \pm 31.8$; $F(1, 44) = 6.66$, $p = .01$, $d = .77$).

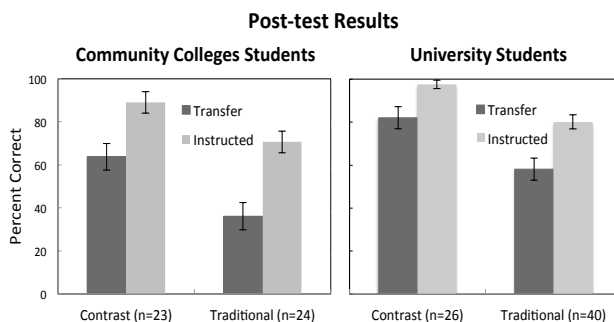


Figure 2. Post-test performance (adjusted percent correct) as a function of condition and problem type in community-college students and university students.

Although pretest score was related to posttest performance ($F(2, 62) = 25.1$, $p < .001$) in the university students, a similar pattern of results for posttest performance was obtained (see right panel of Figure 2). Controlling for pretest score, the causal-contrast group substantially outperformed the traditional group on both the transfer problems ($M_{contrast} = 82.0 \pm 25.9$, $M_{traditional} = 58.1 \pm 32.3$; $F(1, 63) = 16.53$, $p < .001$, $d = .61$) and the instructed problems ($M_{contrast} = 97.6 \pm 10.3$, $M_{traditional} = 80.1 \pm 21.4$; $F(1, 63) = 23.04$, $p < .001$, $d = .88$).

Looking more closely at the performance of the community-college students, we can see that in transfer Problem 6 in Figure 3, there was a large difference in performance across conditions: 62.5% correct versus 39.1% respectively for the causal-contrast and traditional groups,

$\chi^2(1, N=47) = 2.57$, $p = .05$. Recall that this problem is especially informative. The superior performance of the causal-contrast participants indicates that their intervention did not mislead them into formulating a simplistic rule regarding the joint presence of x and x^2 that ignores whether they are terms or factors. Instead, the intervention enhanced the correct flexible use of the concepts and procedures.

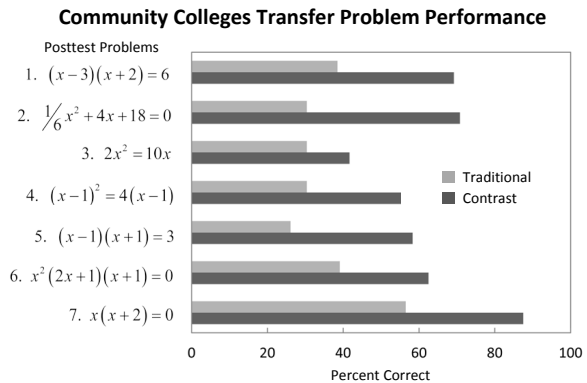


Figure 3. Performance on individual transfer problems by community-college traditional and causal-contrast groups.

Problem 5 in Figure 3 shows another large between-group difference in performance: 58.3% versus 26.1% respectively for causal-contrast and traditional, $\chi^2(1, N=47) = 5.0$, $p = .01$. The successful students expanded the LHS and rearranged the equation to obtain a 0 on the RHS. Most traditional subjects made the common error discussed earlier.

To further assess the acquisition of goal-directed reasoning, we examined participants' solutions to two types of posttest problems. *Expansion* problems are factored equations that must be expanded and rearranged in order to apply the ZPP (e.g., see Problem 5 in Figure 3). In contrast, the ZPP can be directly applied to *non-expansion* problems (e.g., Problem 6 in Figure 3)—that is, there is no need to perform an additional step such as expanding the LHS to enable the application of the ZPP. Novices, however, often circuitously expand the LHS and simplify it back to its original form before applying the ZPP. Participants were coded as goal-directed if they solved all expansion problems and solved a majority (2 out of 3) of the non-expansion problems without the unnecessary expansion. We found that for the community-college students, 54.2% of the causal-contrast group were goal-directed, compared to only 21.7% of the traditional group, $\chi^2(1, N=47) = 3.94$, $p = .047$. A similar pattern was found in the university students (53.8% vs 27.5%), $\chi^2(1, N=66) = 3.60$, $p = .058$.

The consistent difference in performance between conditions on each transfer problem (see Figure 3) combined with the qualitative differences across conditions in goal-directed solving indicates that when students understand why they are performing certain operations in a procedure, and what those operations do in the context of the obstacles to be overcome, they are better able to flexibly use an operation to achieve an end, resulting in greater

success on transfer problems. On the other hand, when students merely follow a procedure that they memorized—even if they were once told the reasons for the operations in that procedure—they are not as proficient in flexibly using the operations to solve novel problems; their representation of the underlying causal structure is evidently incomplete.

Discussion

Our results show that relative to a traditional approach, the causal-contrast approach dramatically improves algebra generalization and transfer in both community-college and university students. Regardless of prior knowledge, the causal-contrast participants showed both an enhanced ability to solve novel transfer problems and a tendency for goal-directed solving. These results support the causal contrast hypothesis that the combination of failures and conditional-contingency information directs attention to the relevant causal relations in the solution structure, enhancing students' ability to decompose a procedure and use the constituent operations flexibly in solving problems.

Our findings are particularly striking in view of students' typical inability to solve novel problems; the issue of knowledge transfer is so pervasive, in fact, that mathematics educators have termed it the "inert knowledge problem". Our results show that even when traditional, analytic instruction focuses on teaching the reasons for mathematical procedures, students still fail to learn in a way that promotes generalization to novel problems. In contrast, by allowing students to use an implicit, empirical learning process to *discover* the causal structure of solutions, students with different levels of skill and knowledge are able to fill in the missing links of their causal structure, and consequently need not rely on their rote memory of procedures.

Acknowledgments

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