Analogy and Arithmetic: An HDTP-Based Model of the Calculation Circular Staircase

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Abstract

Analogical reasoning and its applications are gaining attention not only in cognitive science but also in the context of education and teaching. In this paper we provide a short analysis and a detailed formal model (based on the Heuristic-Driven Theory Projection framework for computational analogy-making) of the Calculation Circular Staircase, a tool for teaching basic arithmetic and insights based on the ordinal number conception of the natural numbers to children in their first years of primary school. We argue that such formal methods and computational accounts of analogy-making can be used to gain additional insights in the inner workings of analogy-based educational methods and tools.
Keywords: Analogy, Education, Teaching, Arithmetic, Formal Model, Computational Analogy-Making, HDTP.

Introduction

Analogical reasoning is the ability to perceive, and operate on, dissimilar domains as similar with respect to certain aspects based on shared commonalities in relational structure or appearance. This has been proposed as an essential aspect of the ability to learn abstract concepts or procedures (Gentner, Holyoak, & Kokinov, 2001), and is recognised as ubiquitous in human reasoning and problem solving (Gentner, 1983), representational transfer (Novick, 1988), and adaptation to novel contexts (Holyoak & Thagard, 1995).

Inherent in the structure of analogical reasoning is its role in education and learning: new ideas can be constructed and explored in relation to familiar concepts. While substantial research has been carried out into the role of analogical reasoning and science education (see, for instance, (Duit, 1991; Arnold & Millar, 1996; Guerra-Ramos, 2011)), its role in mathematics education has been somewhat less explored – although notable exceptions include (Pimm, 1981; English, 1997). These studies support our assumption that analogies can be used for facilitating the understanding of concepts and procedures in abstract and formal domains, such as mathematics, physics or science. The pedagogical use of analogies as a means of triggering, framing and guiding creative insight processes still needs to be widely recognised as part of teaching expertise and incorporated into innovative teacher education schemes (Akgul, 2006).

In this paper, we want to contribute to a deeper understanding of the role and the mode of operation of analogy in an educational context by first providing a description and short analysis of the analogy-based Calculation Circular Staircase used for teaching basic arithmetic to children attending their initial mathematics classes at the beginning of primary school (Schwank, 2003; Schwank, Aring, & Blocksdorf, 2005), before showing how a computational analogy-making framework as Heuristic-Driven Theory Projection (HDTP) (Schwering, Krumnack, Kühnberger, & Gust, 2009) can be used to provide a formal computational reconstruction of the staircase as a prototypical example of analogy-use taken from a real-life teaching situation. We thereby also continue the work started in (Besold, 2013) with a far more complex and deep-rooted case study. By doing so, we aim to show one way (amongst several) of how analogy-engines and their corresponding background theories can fruitfully be applied to modeling and analysis tasks from the field of psychology of learning, education, and didactics.

Heuristic-Driven Theory Projection (HDTP)

There is much work on both theoretical and computational models of analogy-making. Heuristic-Driven Theory Projection (HDTP) (Schwering et al., 2009) is one such perspective: this is a formal theory and corresponding software implementation, conceived as a mathematically sound framework for analogy-making. HDTP has been created for computing analogical relations and inferences for domains which are given in the form of a many-sorted first-order logic representation. Source and target of the analogy-making process are defined in terms of axiomatizations, i.e., given by a finite set of formulae. HDTP tries to produce a generalization of both domains by aligning pairs of formulae from the two domains by means of anti-unification: Anti-unification tries to solve the problem of generalizing terms in a meaningful way, yielding for each term an anti-instance, in which distinct subterms have been replaced by variables (which in turn would allow for a retrieval of the original terms by a substitution of the variables by appropriate subterms).

HDTP in its present version uses a restricted form of higher-order anti-unification. In higher-order anti-unification, classical first-order terms are extended by the introduction of variables which may take arguments (where classical first-order variables correspond to variables with arity 0), making a term either a first-order or a higher-order term. Then, anti-unification can be applied analogously to the original first-order case, yielding a generalization subsuming the specific terms. The class of substitutions which are applicable in HDTP is restricted to (compositions of) the following four cases: renamings (replacing a variable by another variable of the same argument structure), fixations (replacing a variable by a function symbol of the same argument structure), ar-

Figure 1: The "big" Calculation Circular Staircase (as depicted in (Schwank et al., 2005)): Numbers from 1 to 9 are represented by orange balls in the inner circle, numbers from 10 to 19 by green and orange balls in the outer one, the white door on the right marks the transition point between circles.

gument insertions, and permutations (an operation rearranging the arguments of a term). This formalism has proven capable of detecting structural commonalities not accessible to first-order anti-unification, as for instance also structural commonalities between functions and predicates within the logical language can be found and exploited (whilst the first-order formalism would in these be limited to the respective argument positions only), allowing for a better recognition of relational mappings (as opposed to mere attribute mappings). Once the generalization has been computed, the alignments of formulae together with the respective generalizations can be read as proposals of analogical relations between source and target domain, and can be used for guiding an analogy-based process of transferring knowledge between both domains. Analogical transfer results in structure enrichment on the target side, which corresponds to the addition of new axioms to the target theory, but may also involve the addition of new first-order symbols.

The Calculation Circular Staircase

Dedekind (Dedekind, 1887/1969) argued that ordinal numbers and insights into the basic structure of the natural numbers play a crucial role in understanding (and thus also teaching) the foundations of arithmetic. About a century later, studies by Brainerd (Brainerd, 1979) also showed that a deeper understanding of the ordinals supports and facilitates the learning of basic arithmetic operations in children. Based on this line of thought, the Calculation Circular Staircase (Schwank, 2003; Schwank et al., 2005) has been developed.

A Teaching Tool for Basic Arithmetic

Learning by analogy requires conceiving of and performing a transfer mapping of concepts and relational structures from a better-known base domain into a less familiar target domain. This mapping is typically established by a pairwise matching of individual elements from the respective domains, resulting in a set of systematic correspondences. In the context of mathematics education and mathematical reasoning, children are required to understand abstract relations and operations (such as equality, addition, and subtraction) which can best be taught by drawing parallels between similar examples in less abstract domains (Clement, 1993). Still, the availability of supportive cues for the analogy is crucial for the success of the learning process (Glynn, Duit, & Thiele, 1995).

The Calculation Circular Staircase (cf. Fig. 1) offers children a means of developing an understanding of the interpretation of numbers as results of transformation operations. This goal shall be achieved by enabling a mental functional motor skill-based way of accessing the foundational construction principles of the number space and the corresponding basic arithmetic operations: The numbers from 0 to 9 are represented in the inner circle by a corresponding number of orange balls, numbers from 10 to 19 are represented in the outer circle at the respective places (corresponding to the inner circle's ordering) by 10 green balls and a corresponding number of orange ones. A little door indicates the point of transition between circles. Arithmetic operations are introduced via "magical" signs (showing "+" or "-") carried by toy figures. When equipped with the respective sign, a toy figure can perform jumps on the staircase – before moving, a decision for a sign has to be taken (involving the child in a responsible and motor active way instead of assigning the role of a passive spectator). Addition corresponds to an ascending movement, subtraction to a movement in descending direction. This enables children to experience subtraction as a proper inverse operation to addition, arising naturally from the "wish" of a toy figure to also descend the staircase.

Decimal structure-based analogies between different computations (e.g. between "$5-4=1$" and "$15-4=11$") are made accessible to children's understanding via synchronous movements of two toy figures in the inner and outer circle, respectively: The 10 green balls in the outer circle stay constant, with respect to the orange balls identical movements yield identical results. The door between the circles provides children with a natural "resting point" for simplifying difficult computations involving a decimal transition: If e.g. $13-5$ shall be computed, the toy figure (carrying the "-" sign) first moves to the column representing the number 10 (being the only column in the outer circle not containing an orange ball and being directly next to the door, naturally corresponding to the 0-column in the inner circle), losing as many orange balls in height as were initially situated below the figure. The remaining height difference of 2 is now accounted for in a second step, thus transforming the original task into the easy to handle $10-2$ and making the decimal transition attractive for the children. Also, the number 0 obtains a natural position in the number system of the Calculation Circular Staircase, simply corresponding to the result of performing another step down from the first stair. Having the caesura between the two circles after the representation of the number 9 becomes meaningful to the children once they write down the corresponding numbers, there also encountering a significant difference between the initial one-digit numbers and the two-digit numbers starting with 10.

At the level of cognitive analysis, the idea underlying the Calculation Circular Staircase is an active recruitment of the "functional thinking" approach to mathematics (cf., e.g., (Schwank, Gelfman, & Nardi, 1999)). As opposed to a "predicative thinking"-style understanding of relationships within mathematics, which uses equality as ordering principle when conceptualizing mathematical structures (i.e., mathematics being conceptualized on the basis of the repeated applicability of certain predicates), the "functional thinking" perspective bases its conceptualizations on differences between mathematical concepts which can then be used to conceive of a construction process for the respective class of structures (i.e., mathematics being conceptualized on the basis of repeated constructive steps). In their interaction with the Calculation Circular Staircase, the children are naturally led into taking the "functional thinking" approach across repeated stages of play, each time actively becoming aware of the individual steps a toy figure has to take when changing from one position on the staircase to another one. Instead of *post hoc* merely checking whether a certain distance has been covered between the initial step and the final one (i.e., whether a predicate indicating a certain value for the difference between both steps holds), they experience an active construction process explicating the guided transition from the initial step to the final one.

An HDTP-Based Model of the Staircase

We now reconstruct the "big" Calculation Circular Staircase (i.e., the version equipped with two circles or 19 steps) as an analogy-based model for understanding and learning among others important aspects of the ordinal number conception of the natural numbers in the range from 0 to 19.

The analogy uses the Calculation Circular Staircase as a base domain, transferring the structure and relational conception children acquire by playing with the staircase into their previously acquired knowledge about natural numbers as target domain. The latter domain is most likely initially still very poor as compared to the Calculation Circular Staircase domain as only very little (if any at all) internal structure or relations have been acquired besides the mere ordering of the number terms from one to nineteen that had been committed to memory in previous lessons. And even for this ordering it can be assumed that the ordering has mostly only been developed on basis of isolated neighboring tuples of the form $(n, n+1)$, for each number term only remembering its immediate successor. The arithmetic operations "+" (addition) and "−" (subtraction) are known as abstract concepts (as are their corresponding addition and subtraction tables), but have not yet been developed into a grounded, constructively applicable conceptualization. Table 1 gives a formal HDTP-style model of the Calculation Circular Staircase, whilst an idealized version (i.e., a version featuring complete addition and subtraction tables, which in reality should be assumed to be rather incomplete or sparse) of the students' initial conceptualization of the natural number domain can formally be represented as shown in Table 2.

Sorts:
steps, sign, circle, caesura, direction, time, natural.
Entities:
$\mathit{one}, \mathit{two}, \mathit{three}, \ldots, \mathit{nineteen}, \mathit{zero}, S_a, S_b, S_c, S_d : \mathit{steps}.$
$+, - : \mathit{sign}.$
$\mathit{up}, \mathit{down}, D_1 : \mathit{direction}.$
$\mathit{door}, C_1 : \mathit{caesura}.$
$T_1, T_2 : \mathit{time}.$
$\mathit{inner}, \mathit{outer}, Ci_1 : \mathit{circle}.$
Functions:
$\mathit{height} : \mathit{steps} \to \mathit{natural}.$
$\mathit{interpretation} : \mathit{sign} \to \mathit{direction}.$
$\mathit{dist} : \mathit{direction} \times \mathit{steps} \times \mathit{steps} \to \mathit{direction} \times \mathit{natural}.$
$\mathit{magn} : \mathit{direction} \times \mathit{natural} \to \mathit{natural}.$
$\mathit{diff} : \mathit{direction} \times \mathit{steps} \times \mathit{steps} \to \mathit{natural}.$
Predicates:
$\mathit{succ} : \mathit{steps} \times \mathit{steps}.$
$\mathit{higher} : \mathit{steps} \times \mathit{steps}.$
$\mathit{lower} : \mathit{steps} \times \mathit{steps}.$
$\mathit{inCircle} : \mathit{circle} \times \mathit{steps}.$
$\mathit{base} : \mathit{steps}.$
$\mathit{inFocus} : \mathit{steps} \times \mathit{time}.$
$\mathit{currentSign} : \mathit{sign} \times \mathit{time}.$
$\mathit{between} : \mathit{caesura} \times \mathit{steps} \times \mathit{steps}.$
$\mathit{move} : \mathit{steps} \times \mathit{natural} \times \mathit{direction} \times \mathit{time}.$
$\mathit{analogs} : \mathit{steps} \times \mathit{steps}.$
Facts:
(s_1) $\mathit{succ}(\mathit{zero}, \mathit{one}).$
(s_2) $\mathit{succ}(\mathit{one}, \mathit{two}).$
$\ldots$
(s_{19}) $\mathit{succ}(\mathit{eighteen}, \mathit{nineteen}).$
(s_{20}) $\mathit{inCircle}(\mathit{inner}, \mathit{zero}).$
(s_{21}) $\mathit{inCircle}(\mathit{inner}, \mathit{one}).$
$\ldots$
(s_{29}) $\mathit{inCircle}(\mathit{inner}, \mathit{nine}).$
(s_{30}) $\mathit{inCircle}(\mathit{outer}, \mathit{ten}).$
$\ldots$
(s_{39}) $\mathit{inCircle}(\mathit{outer}, \mathit{nineteen}).$
(s_{40}) $\mathit{between}(\mathit{door}, \mathit{nine}, \mathit{ten}).$
Laws:
(s_{41}) $\mathit{interpretation}(+) = \mathit{up}.$
(s_{42}) $\mathit{interpretation}(-) = \mathit{down}.$
(s_{43}) $\mathit{interpretation}(+) \neq \mathit{interpretation}(-).$
(s_{44}) $\mathit{higher}(S_a, S_b) \leftrightarrow \mathit{succ}(S_b, S_a) \vee \exists S_c : (\mathit{succ}(S_b, S_c) \wedge \mathit{higher}(S_a, S_c)).$
(s_{45}) $\mathit{higher}(S_a, S_b) \leftrightarrow \mathit{lower}(S_b, S_a).$
(s_{46}) $T_1 < T_2 : \mathit{inFocus}(S_a, T_1) \wedge \mathit{currentSign}(+, T_1) \wedge \mathit{move}(S_a, 1, \mathit{interpretation}(+), T_1) \wedge \mathit{succ}(S_a, S_b) \to \mathit{inFocus}(S_b, T_2) \wedge \mathit{dist}(\mathit{interpretation}(+), S_a, S_b) = (\mathit{interpretation}(+), 1).$
(s_{47}) $T_1 < T_2, \forall n \in \mathbb{N}, n > 1 : \mathit{inFocus}(S_a, T_1) \wedge \mathit{currentSign}(+, T_1) \wedge \mathit{move}(S_a, n, \mathit{interpretation}(+), T_1) \wedge \mathit{succ}(S_a, S_b) \to \mathit{inFocus}(S_b, T_2) \wedge \mathit{currentSign}(+, T_2) \wedge \mathit{move}(S_b, n-1, \mathit{interpretation}(+), T_2) \wedge \mathit{dist}(\mathit{interpretation}(+), S_a, S_b) = (\mathit{interpretation}(+), 1).$
(s_{48}) $T_1 < T_2 : \mathit{inFocus}(S_a, T_1) \wedge \mathit{currentSign}(-, T_1) \wedge \mathit{move}(S_a, 1, \mathit{interpretation}(-), T_1) \wedge \mathit{succ}(S_b, S_a) \to \mathit{inFocus}(S_b, T_2) \wedge \mathit{dist}(\mathit{interpretation}(-), S_a, S_b) = (\mathit{interpretation}(-), 1).$
(s_{49}) $T_1 < T_2, \forall n \in \mathbb{N}, n > 1 : \mathit{inFocus}(S_a, T_1) \wedge \mathit{currentSign}(-, T_1) \wedge \mathit{move}(S_a, n, \mathit{interpretation}(-), T_1) \wedge \mathit{succ}(S_b, S_a) \to \mathit{inFocus}(S_b, T_2) \wedge \mathit{currentSign}(-, T_2) \wedge \mathit{move}(S_b, n-1, \mathit{interpretation}(-), T_2).$
(s_{50}) $\mathit{lower}(S_a, S_b), T_1 < T_2, \forall n \in \mathbb{N} : \mathit{dist}(\mathit{interpretation}(+), S_a, S_b) = (\mathit{interpretation}(+), n) \wedge \mathit{inFocus}(S_b, T_1) \wedge \mathit{currentSign}(+, T_1) \wedge \mathit{move}(S_b, 1, \mathit{interpretation}(+), T_1) \wedge \mathit{succ}(S_b, S_c) \to \mathit{inFocus}(S_c, T_2) \wedge \mathit{dist}(\mathit{interpretation}(+), S_a, S_c) = (\mathit{interpretation}(+), n+1).$
(s_{51}) $\mathit{higher}(S_a, S_b), T_1 < T_2, \forall n \in \mathbb{N} : \mathit{dist}(\mathit{interpretation}(-), S_a, S_b) = (\mathit{interpretation}(-), n) \wedge \mathit{inFocus}(S_b, T_1) \wedge \mathit{currentSign}(-, T_1) \wedge \mathit{move}(S_b, 1, \mathit{interpretation}(-), T_1) \wedge \mathit{succ}(S_c, S_b) \to \mathit{inFocus}(S_c, T_2) \wedge \mathit{dist}(\mathit{interpretation}(-), S_a, S_c) = (\mathit{interpretation}(-), n+1).$
(s_{52}) $\forall n \in \mathbb{N} : \mathit{magn}(D_1, n) \to n.$
(s_{53}) $\mathit{lower}(S_a, S_b) : \mathit{diff}(\mathit{interpretation}(+), S_a, S_b) = \mathit{diff}(\mathit{interpretation}(-), S_b, S_a) = \mathit{magn}(\mathit{dist}(\mathit{interpretation}(+), S_a, S_b)).$
(s_{54}) $\nexists S_a : \mathit{lower}(S_a, S_b) \to \mathit{height}(S_c) = \mathit{diff}(\mathit{interpretation}(-), S_c, S_b).$
(s_{55}) $\mathit{between}(C_1, S_a, S_b) \vee (\mathit{inCircle}(Ci_1, S_b) \wedge \nexists S_c : (\mathit{inCircle}(Ci_1, S_c) \wedge \mathit{lower}(S_c, S_b))) \to \mathit{base}(S_b).$
(s_{56}) $S_a \neq S_b \wedge \exists S_c, S_d : (\mathit{base}(S_c) \wedge \mathit{base}(S_d) \wedge \mathit{dist}(\mathit{interpretation}(-), S_a, S_c) = \mathit{dist}(\mathit{interpretation}(-), S_b, S_d)) \to \mathit{analogs}(S_a, S_b).$

Table 1: Formalization of the Calculation Circular Staircase.

We quickly want to focus on some aspects of the respective formalizations. The base domain of the later analogy, i.e., the formalization of the Calculation Circular Staircase,

Sorts:
number, *sign*, *operation*.
Entities:
one, *two*, …, *nineteen*, *zero* : *number*.
$+, -$: *sign*.
plus, *minus* : *operation*.
Functions:
$\mathit{apply} : \mathit{operation} \times \mathit{number} \times \mathit{number} \to \mathit{number}$.
$\mathit{interpretation} : \mathit{sign} \to \mathit{operation}$.
Predicates:
$\mathit{succ} : \mathit{number} \times \mathit{number}$.
Facts:
(n_1) $\mathit{succ}(\mathit{zero}, \mathit{one})$.
(n_2) $\mathit{succ}(\mathit{one}, \mathit{two})$.
(n_3) $\mathit{succ}(\mathit{two}, \mathit{three})$.
(n_4) $\mathit{succ}(\mathit{three}, \mathit{four})$.
…
(n_{19}) $\mathit{succ}(\mathit{eighteen}, \mathit{nineteen})$.
(n_{20}) $\mathit{apply}(\mathit{interpretation}(+), \mathit{one}, \mathit{one}) = \mathit{two}$.
(n_{21}) $\mathit{apply}(\mathit{interpretation}(+), \mathit{one}, \mathit{two}) = \mathit{three}$.
(n_{22}) $\mathit{apply}(\mathit{interpretation}(+), \mathit{one}, \mathit{three}) = \mathit{four}$.
…
(n_{37}) $\mathit{apply}(\mathit{interpretation}(+), \mathit{one}, \mathit{eighteen}) = \mathit{nineteen}$.
(n_{38}) $\mathit{apply}(\mathit{interpretation}(+), \mathit{two}, \mathit{one}) = \mathit{three}$.
(n_{39}) $\mathit{apply}(\mathit{interpretation}(+), \mathit{two}, \mathit{two}) = \mathit{four}$.
…
(n_{54}) $\mathit{apply}(\mathit{interpretation}(+), \mathit{two}, \mathit{seventeen}) = \mathit{nineteen}$.
(n_{55}) $\mathit{apply}(\mathit{interpretation}(+), \mathit{three}, \mathit{one}) = \mathit{four}$.
…
(n_{190}) $\mathit{apply}(\mathit{interpretation}(+), \mathit{eighteen}, \mathit{one}) = \mathit{nineteen}$.
(n_{191}) $\mathit{apply}(\mathit{interpretation}(-), \mathit{two}, \mathit{one}) = \mathit{one}$.
(n_{192}) $\mathit{apply}(\mathit{interpretation}(-), \mathit{three}, \mathit{one}) = \mathit{two}$.
(n_{193}) $\mathit{apply}(\mathit{interpretation}(-), \mathit{three}, \mathit{two}) = \mathit{one}$.
…
(n_{360}) $\mathit{apply}(\mathit{interpretation}(-), \mathit{nineteen}, \mathit{seventeen}) = \mathit{two}$.
(n_{361}) $\mathit{apply}(\mathit{interpretation}(-), \mathit{nineteen}, \mathit{eighteen}) = \mathit{one}$.
Laws:
(n_{362}) $\mathit{interpretation}(+) = \mathit{plus}$.
(n_{363}) $\mathit{interpretation}(-) = \mathit{minus}$.
(n_{364}) $\mathit{interpretation}(+) \neq \mathit{interpretation}(-)$.

Table 2: Formalization of an idealized form of the children's initial conception of the number domain.

exhibits a rich structure, both concerning facts and laws alike. The facts represent the easily accessible structure of the staircase, namely the order of succession of the steps, the distinction between the inner and the outer circle, and the placement of the door between steps nine and ten. The laws cover the transformational and constructive insights accessible to the children via interaction with the staircase: For instance (s_{46}) to (s_{49}) encompass the previously described process of having the toy figure move up or down the staircase, and (s_{50}) and (s_{51}) then add a counting principle keeping track of the number of steps passed by the figure (which in reality allows children to determine the distance the toy figure may move on the staircase). (s_{52}) and (s_{53}) serve for converting the distance measured in steps into a natural number, i.e., represent the children's mental process when realizing that distances on the staircase correspond to a more abstract number concept, i.e., are not bound to the individual stairs but can be generalized. Concerning the final two laws, (s_{55}) introduces the previously mentioned concept of singular steps in the Calculation Circular Staircase which are similar in that they form the base of one closed part of the staircase (namely of the inner or outer circle) or are marked by being preceded by the door as caesura, and (s_{56}) concludingly introduces the concept of structure-based analogs amongst the steps.

The formalization of the target domain of the later analogy, i.e., of an idealized version of the children's initial conception of the number domain, contains mostly facts the children have learned by heart, namely the order of the number terms

Sorts:
circle, *caesura*, *time*, *sign*, *direction/operation*, *steps/number*, *natural/number*.
Entities:
one, *two*, …, *nineteen*, *zero*, S_a, S_b, S_c, S_d : *steps/number*.
O : *direction/operation*.
$+, -$: *sign*.
$(*)$ *door*, C_1 : *caesura*.
$(*)$ T_1, T_2 : *time*.
$(*)$ *inner*, *outer*, Ci_1 : *circle*.
Functions:
$\mathit{DiffApply} : \mathit{direction/operation} \times \mathit{steps/number} \times \mathit{steps/number} \to \mathit{natural/number}$.
$\mathit{interpretation} : \mathit{sign} \to \mathit{direction/operation}$.
$(*)$ $\mathit{height} : \mathit{steps/number} \to \mathit{natural/number}$.
$(*)$ $\mathit{dist} : \mathit{direction/operation} \times \mathit{steps/number} \times \mathit{steps/number} \to \mathit{direction/operation} \times \mathit{natural/number}$.
$(*)$ $\mathit{magn} : \mathit{direction/operation} \times \mathit{natural/number} \to \mathit{natural/number}$.
Predicates:
$\mathit{succ} : \mathit{steps/number} \times \mathit{steps/number}$.
$(*)$ $\mathit{higher} : \mathit{steps/number} \times \mathit{steps/number}$.
$(*)$ $\mathit{lower} : \mathit{steps/number} \times \mathit{steps/number}$.
$(*)$ $\mathit{inCircle} : \mathit{circle} \times \mathit{steps/number}$.
$(*)$ $\mathit{base} : \mathit{steps/number}$.
$(*)$ $\mathit{inFocus} : \mathit{steps/number} \times \mathit{time}$.
$(*)$ $\mathit{currentSign} : \mathit{sign} \times \mathit{time}$.
$(*)$ $\mathit{between} : \mathit{caesura} \times \mathit{steps/number} \times \mathit{steps/number}$.
$(*)$ $\mathit{move} : \mathit{steps/number} \times \mathit{natural/number} \times \mathit{direction/operation} \times \mathit{time}$.
$(*)$ $\mathit{analogs} : \mathit{steps/number} \times \mathit{steps/number}$.
Facts:
(g_1) $\mathit{succ}(\mathit{zero}, \mathit{one})$.
(g_2) $\mathit{succ}(\mathit{one}, \mathit{two})$.
…
(g_{19}) $\mathit{succ}(\mathit{eighteen}, \mathit{nineteen})$.
$(g_{20}*)$ $\mathit{inCircle}(\mathit{inner}, \mathit{zero})$.
$(g_{21}*)$ $\mathit{inCircle}(\mathit{inner}, \mathit{one})$.
…
$(g_{29}*)$ $\mathit{inCircle}(\mathit{inner}, \mathit{nine})$.
$(g_{30}*)$ $\mathit{inCircle}(\mathit{outer}, \mathit{ten})$.
…
$(g_{39}*)$ $\mathit{inCircle}(\mathit{outer}, \mathit{nineteen})$.
$(g_{40}*)$ $\mathit{between}(\mathit{door}, \mathit{nine}, \mathit{ten})$.
Laws:
(g_{41}) $\mathit{interpretation}(+) = O$.
(g_{42}) $\mathit{interpretation}(-) = O$.
(g_{43}) $\mathit{interpretation}(+) \neq \mathit{interpretation}(-)$.
$(g_{44}*)$ $\mathit{higher}(S_a, S_b) \leftrightarrow \mathit{succ}(S_b, S_a) \vee \exists S_c : (\mathit{succ}(S_b, S_c) \wedge \mathit{higher}(S_a, S_c))$.
$(g_{45}*)$ $\mathit{higher}(S_a, S_b) \leftrightarrow \mathit{lower}(S_b, S_a)$.
$(g_{46}*)$ $T_1 < T_2 : \mathit{inFocus}(S_a, T_1) \wedge \mathit{currentSign}(+, T_1) \wedge \mathit{move}(S_a, 1, \mathit{interpretation}(+), T_1) \wedge \mathit{succ}(S_a, S_b) \to \mathit{inFocus}(S_b, T_2) \wedge \mathit{dist}(\mathit{interpretation}(+), S_a, S_b) = (\mathit{interpretation}(+), 1)$.
$(g_{47}*)$ $T_1 < T_2, \forall n \in \mathbb{N}, n > 1 : \mathit{inFocus}(S_a, T_1) \wedge \mathit{currentSign}(+, T_1) \wedge \mathit{move}(S_a, n, \mathit{interpretation}(+), T_1) \wedge \mathit{succ}(S_a, S_b) \to \mathit{inFocus}(S_b, T_2) \wedge \mathit{currentSign}(+, T_2) \wedge \mathit{move}(S_b, n-1, \mathit{interpretation}(+), T_2) \wedge \mathit{dist}(\mathit{interpretation}(+), S_a, S_b) = (\mathit{interpretation}(+), 1)$.
$(g_{48}*)$ $T_1 < T_2 : \mathit{inFocus}(S_a, T_1) \wedge \mathit{currentSign}(-, T_1) \wedge \mathit{move}(S_a, 1, \mathit{interpretation}(-), T_1) \wedge \mathit{succ}(S_b, S_a) \to \mathit{inFocus}(S_b, T_2) \wedge \mathit{dist}(\mathit{interpretation}(-), S_a, S_b) = (\mathit{interpretation}(-), 1)$.
$(g_{49}*)$ $T_1 < T_2, \forall n \in \mathbb{N}, n > 1 : \mathit{inFocus}(S_a, T_1) \wedge \mathit{currentSign}(-, T_1) \wedge \mathit{move}(S_a, n, \mathit{interpretation}(-), T_1) \wedge \mathit{succ}(S_b, S_a) \to \mathit{inFocus}(S_b, T_2) \wedge \mathit{currentSign}(-, T_2) \wedge \mathit{move}(S_b, n-1, \mathit{interpretation}(-), T_2)$.
$(g_{50}*)$ $\mathit{lower}(S_a, S_b), T_1 < T_2, \forall n \in \mathbb{N} : \mathit{dist}(\mathit{interpretation}(+), S_a, S_b) = (\mathit{interpretation}(+), n) \wedge \mathit{inFocus}(S_b, T_1) \wedge \mathit{currentSign}(+, T_1) \wedge \mathit{move}(S_b, 1, \mathit{interpretation}(+), T_1) \wedge \mathit{succ}(S_b, S_c) \to \mathit{inFocus}(S_c, T_2) \wedge \mathit{dist}(\mathit{interpretation}(+), S_a, S_c) = (\mathit{interpretation}(+), n+1)$.
$(g_{51}*)$ $\mathit{higher}(S_a, S_b), T_1 < T_2, \forall n \in \mathbb{N} : \mathit{dist}(\mathit{interpretation}(-), S_a, S_b) = (\mathit{interpretation}(-), n) \wedge \mathit{inFocus}(S_b, T_1) \wedge \mathit{currentSign}(-, T_1) \wedge \mathit{move}(S_b, 1, \mathit{interpretation}(-), T_1) \wedge \mathit{succ}(S_c, S_b) \to \mathit{inFocus}(S_c, T_2) \wedge \mathit{dist}(\mathit{interpretation}(-), S_a, S_c) = (\mathit{interpretation}(-), n+1)$.
$(g_{52}*)$ $\forall n \in \mathbb{N} : \mathit{magn}(D_1, n) \to n$.
$(g_{53}*)$ $\mathit{lower}(S_a, S_b) : \mathit{DiffApply}(\mathit{interpretation}(+), S_a, S_b) = \mathit{DiffApply}(\mathit{interpretation}(-), S_b, S_a) = \mathit{magn}(\mathit{dist}(\mathit{interpretation}(+), S_a, S_b))$.
$(g_{54}*)$ $\nexists S_a : \mathit{lower}(S_a, S_b) \to \mathit{height}(S_c) = \mathit{DiffApply}(\mathit{interpretation}(-), S_c, S_b)$.
$(g_{55}*)$ $\mathit{between}(C_1, S_a, S_b) \vee (\mathit{inCircle}(Ci_1, S_b) \wedge \nexists S_c : (\mathit{inCircle}(Ci_1, S_c) \wedge \mathit{lower}(S_c, S_b))) \to \mathit{base}(S_b)$.
$(g_{56}*)$ $S_a \neq S_b \wedge \exists S_c, S_d : (\mathit{base}(S_c) \wedge \mathit{base}(S_d) \wedge \mathit{dist}(\mathit{interpretation}(-), S_a, S_c) = \mathit{dist}(\mathit{interpretation}(-), S_b, S_d)) \to \mathit{analogs}(S_a, S_b)$.

Table 3: Generalized theory of the Calculation Circular Staircase and the children's number domain, already expanded by the generalized forms of the candidate elements for analogical transfer from base to target domain (marked with $*$).

between zero and nineteen, and addition and subtraction tables within this range. In reality it has to be assumed that the addition and subtraction tables are significantly more sparsely populated than in our formalization, corresponding to incomplete recall of the memorized full tables.

The HDTP mechanism can now be used for computing a common generalization of both domains, yielding a generalized theory like given in Table 3. The main domain elements defining the alignment of formulae are the matching between the entities of sort *steps* and *number*, between the functions *diff* and *apply*, the alignment of the respective *sign* entities, as well as the matching between the *direction* and *operation* entities (induced by the alignment of the respective *interpretation* functions). Here it has to be noted that in order to analogically match the two domains it is not only necessary to generalize facts and laws but in this case also sorts have to be generalized, for two sorts yielding the least general supersort. This is needed, for instance, when pairing up the representation of the staircase's steps (conceived as mere pillars) and the number terms known to the children.

In conclusion, the generalized theory forms the basis for transferring knowledge in an analogy-based way from the (originally richer) Calculation Circular Staircase domain to the children's number domain, resulting in an expanded theory for the numbers as given in Table 4. The important aspect in this expanded version of the domain is the availability of the constructive relations and insights obtained in the interaction with the Calculation Circular Staircase, e.g., providing a means to give meaning to the number terms via the assignment of the corresponding natural number values (using the *diff* function in ($e_{381}*$)) or via laws ($e_{373}*$) to ($e_{378}*$) allowing for the independent computation of parts of the addition and subtraction table that might not be obtainable from memory (i.e., that would not explicitly be present as a fact in a more realistic formalization of the number domain).

Related Work and Conclusion

We are not the first to consider the use of formal models and computational analogy-making systems in the context of education and teaching-related topics. Among many others, for example in (Thagard, Cohen, & Holyoak, 1989), the authors present a theory and implementation of analogical mapping that applies to explanations of unfamiliar phenomena, and (Siegler, 1989) briefly conjectures how the Structure-Mapping Engine (SME) (Falkenhainer, Forbus, & Gentner, 1989) as a prototypical analogy-engine could be used to gain insights about developmental aspects of analogy use. General cognitive theories of analogical reasoning and associated computation models are also highly relevant to analogies as learning mechanisms. These include Gentner's structure mapping theory and engine, in which relations between objects are preserved (and relations which contribute to higher order predicates are mapped preferentially), and attributes of objects are not mapped (Gentner, 1983); Holyoak and Thagard's multi-constraint theory, in which mappings are evaluated according to constraints of structural consistency, pragmatic centrality and semantic similarity (Holyoak & Thagard, 1997); and Hummel and Holyoak's theory of analogy formation, which integrates memory access and structural mapping, implemented in LISA (Hummel & Holyoak, 2003) (see (Gentner et al., 2001) for a review of computational models of

Sorts:

circle, caesura, time, sign, operation, number, natural.

Entities:

$one, two, \ldots, nineteen, zero, S_a, S_b, S_c, S_d : number.$
$+, - : sign.$
$plus, minus : operation.$
$(*)\ door, C_1 : caesura.$
$(*)\ T_1, T_2 : time.$
$(*)\ inner, outer, Ci_1 : circle.$

Functions:

$apply : operation \times number \times number \to number.$
$interpretation : sign \to operation.$
$(*)\ diff : operation \times number \times number \to natural.$
$(*)\ height : number \to natural.$
$(*)\ dist : operation \times number \times number \to operation \times natural.$
$(*)\ magn : operation \times natural \to natural.$
$(*)\ interpretation : sign \to operation.$

Predicates:

$succ : number \times number.$
$(*)\ higher : number \times number.$
$(*)\ lower : number \times number.$
$(*)\ inCircle : circle \times number.$
$(*)\ base : number.$
$(*)\ inFocus : number \times time.$
$(*)\ currentSign : sign \times time.$
$(*)\ between : caesura \times number \times number.$
$(*)\ move : number \times natural \times operation \times time.$
$(*)\ analogs : number \times number.$

Facts:

$(e_1)\ succ(zero, one).$
$(e_2)\ succ(one, two).$
...
$(e_{19})\ succ(eighteen, nineteen).$
$(e_{20})\ apply(interpretation(+), one, one) = two.$
$(e_{21})\ apply(interpretation(+), one, two) = three.$
...
$(e_{190})\ apply(interpretation(+), eighteen, one) = nineteen.$
$(e_{191})\ apply(interpretation(-), two, one) = one.$
$(e_{192})\ apply(interpretation(-), three, one) = two.$
$(e_{193})\ apply(interpretation(-), three, two) = one.$
...
$(e_{360})\ apply(interpretation(-), nineteen, seventeen) = two.$
$(e_{361})\ apply(interpretation(-), nineteen, eighteen) = one.$
$(e_{362}*)\ inCircle(inner, zero).$
$(e_{363}*)\ inCircle(inner, one).$
...
$(e_{364}*)\ inCircle(inner, nine).$
$(e_{365}*)\ inCircle(outer, ten).$
...
$(e_{366}*)\ inCircle(outer, nineteen).$
$(e_{367}*)\ between(door, nine, ten).$

Laws:

$(e_{368})\ interpretation(+) = plus.$
$(e_{369})\ interpretation(-) = minus.$
$(e_{370})\ interpretation(+) \neq interpretation(-).$
$(e_{371}*)\ higher(S_a, S_b) \leftrightarrow succ(S_b, S_a) \vee \exists S_c : (succ(S_b, S_c) \wedge higher(S_a, S_c)).$
$(e_{372}*)\ higher(S_a, S_b) \leftrightarrow lower(S_b, S_a).$
$(e_{373}*)\ T_1 < T_2 : inFocus(S_a, T_1) \wedge currentSign(+, T_1) \wedge move(S_a, 1, interpretation(+), T_1) \wedge succ(S_a, S_b) \to inFocus(S_b, T_2) \wedge dist(interpretation(+), S_a, S_b) = (interpretation(+), 1).$
$(e_{374}*)\ T_1 < T_2, \forall n \in \mathbb{N}, n > 1 : inFocus(S_a, T_1) \wedge currentSign(+, T_1) \wedge move(S_a, n, interpretation(+), T_1) \wedge succ(S_a, S_b) \to inFocus(S_b, T_2) \wedge currentSign(+, T_2) \wedge move(S_b, n-1, interpretation(+), T_2) \wedge dist(interpretation(+), S_a, S_b) = (interpretation(+), 1).$
$(e_{375}*)\ T_1 < T_2 : inFocus(S_a, T_1) \wedge currentSign(-, T_1) \wedge move(S_a, 1, interpretation(-), T_1) \wedge succ(S_b, S_a) \to inFocus(S_b, T_2) \wedge dist(interpretation(-), S_a, S_b) = (interpretation(-), 1).$
$(e_{376}*)\ T_1 < T_2, \forall n \in \mathbb{N}, n > 1 : inFocus(S_a, T_1) \wedge currentSign(-, T_1) \wedge move(S_a, n, interpretation(-), T_1) \wedge succ(S_b, S_a) \to inFocus(S_b, T_2) \wedge currentSign(-, T_2) \wedge move(S_b, n-1, interpretation(-), T_2).$
$(e_{377}*)\ lower(S_a, S_b), T_1 < T_2, \forall n \in \mathbb{N} : dist(interpretation(+), S_a, S_b) = (interpretation(+), n) \wedge inFocus(S_b, T_1) \wedge currentSign(+, T_1) \wedge move(S_b, 1, interpretation(+), T_1) \wedge succ(S_b, S_c) \to inFocus(S_c, T_2) \wedge dist(interpretation(+), S_a, S_c) = (interpretation(+), n+1).$
$(e_{378}*)\ higher(S_a, S_b), T_1 < T_2, \forall n \in \mathbb{N} : dist(interpretation(-), S_a, S_b) = (interpretation(-), n) \wedge inFocus(S_b, T_1) \wedge currentSign(-, T_1) \wedge move(S_b, 1, interpretation(-), T_1) \wedge succ(S_c, S_b) \to inFocus(S_c, T_2) \wedge dist(interpretation(-), S_a, S_c) = (interpretation(-), n+1).$
$(e_{379}*)\ \forall n \in \mathbb{N} : magn(D_1, n) \to n.$
$(e_{380}*)\ lower(S_a, S_b) : diff(interpretation(+), S_a, S_b) = diff(interpretation(-), S_b, S_a) = magn(dist(interpretation(+), S_a, S_b)).$
$(e_{381}*)\ \nexists S_a : lower(S_a, S_b) \to height(S_c) = diff(interpretation(-), S_c, S_b).$
$(e_{382}*)\ between(C_1, S_a, S_b) \vee (inCircle(Ci_1, S_b) \wedge \nexists S_c : (inCircle(Ci_1, S_c) \wedge lower(S_c, S_b))) \to base(S_b).$
$(e_{383}*)\ S_a \neq S_b \wedge \exists S_c, S_d : (base(S_c) \wedge base(S_d) \wedge dist(interpretation(-), S_a, S_c) = dist(interpretation(-), S_b, S_d)) \to analogs(S_a, S_b).$

Table 4: Analogically enriched formalization of the idealized version of the children's conception of the number domain.

analogy). Our HDTP model of the Calculation Circular Staircase is intended to complement such approaches, showing how formal methods and computational accounts of analogy-making can be used to gain additional insights in the inner workings of analogy-based educational methods and tools. By providing a detailed formal description of the involved domains, also sketching how the domains relate to each other in terms of their joint generalization and how this relation can be used to transfer knowledge from the staircase domain into the number domain, we managed to explicate the structural relations and governing laws underlying the Calculation Circular Staircase as teaching model of the natural number domain, and to point out how the identified constructive and transformation-based conceptualizations then also can provide additional support and a deeper-rooted model for the childrens' initially very flat and sparse conception of the number domain.

We see this work as a first step towards the design of analogy-based teaching material, both specifically in arithmetic and, more generally, in mathematics and other disciplines. Modelling educational analogies provides another perspective on a particular analogy, in terms of which information is transferred, what the limitations of the analogy are, or whether it makes unhelpful mappings; and what potential extensions to the analogy it suggests. We envisage that our model of the Calculation Circular Staircase can be used in order to design a lesson plan on the natural number domain. Its usefulness would then be evaluated via empirical studies on the students, testing the depth of their understanding of cardinal and ordinal numbers and basic operations, and whether the students' understanding (and misunderstanding) mirrors the inferences made by the model.

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