New Empirical Tests of a Quantum Model for Question Order Effects

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Abstract

Recent findings show that human inferences and decisions interfere in ways analogous to incompatible quantum observables, and conceptual judgments are inseparable in ways similar to entangled quantum states. This discovery has led a group of physicists and psychologists to form a new field called "quantum cognition," which uses mathematical principles of quantum theory to explain human cognitive behavior. The power of this new theoretical approach is illustrated here by testing an *a priori* and precise prediction derived from quantum theory regarding question order effects commonly observed in survey research. The test of quantum theory was statistically satisfied across a set of 26 national surveys on presidential job approval and country satisfaction in past 10 years. These results suggest that quantum theory, initially invented to explain order effects on measurements in physics, provides a powerful prediction for measurement order effects in social and behavioral sciences too.

The human brain is a powerful and massively complex neural system. It provides the biological substrate for an emergent mind capable of producing highly intelligent cognitive behaviors, such as inferences and decisions. How this happens remains a topic of intense investigation in cognitive neuroscience. The possibility that the brain's tremendous power arises from parallel computations of quantum physical neuronal interactions has been raised (Hameroff & Penrose, 1996; Hagan, Hameroff & Tuszynski, 2002) but strongly criticized (Tegmark, 2000; McKemmish, Reimers, McKenzie, Mark & Hush, 2009). However, what if it is our behavior – rather than our brains – that follows quantum rules?

Supporting this idea, latest evidence shows that human inferences and decisions interfere in ways analogous to incompatible quantum observables (Pothos & Busemeye, 2009; Busemeyer, Wang & Lambert-Mogiliansky, 2008), and conceptual judgments are inseparable in ways similar to entangled quantum states (Aerts & Sozzo, 2011). Formal principles that quantum theorists invented to deal with properties of complex physical systems provide a powerful mathematical description of human behavior (Busemeyer & Bruza, 2012; Khrennikov, 2010). This discovery has led a group of physicists and psychologists to work together and form the new field of "quantum cognition," which uses mathematical principles of quantum theory to explain human cognitive behavior. It has successfully accounted for various puzzling findings in psychological literature, ranging across perception (Atmanspacher, Filk & Romer, 2004), associative memory (Bruza, Kitto, Nelson & McEvoy, 2009), conceptual reasoning (Aerts, 2009), probability judgments (Busemeyer, Pothos, Franco & Trueblood, 2011), decision making (Yukalov & Sornette, 2011), and strategic game behavior (Lambert-Mogiliansky & Busemeyer, 2012). It is plausible that the underlying neural systems follow classical dynamic laws, but the emergent cognitive behaviors are coarse "quantized" descriptions (Atmanspacher & Graben, 2007). In fact, more than half a century ago, founding fathers of quantum theory speculated that fundamental quantum principles have implications outside of physics to human cognitive behavior (Pauli, 1950; Bohr, 1958).

Here we tested a new, *a priori* and precise prediction derived from quantum theory regarding question order effects commonly observed in survey research. This type of exact prediction is rare in social and behavioral sciences. The prediction was statistically supported across a set of 26 national surveys in past 10 years on two important public opinion questions in the U.S.: presidential job approval and country satisfaction. This surprisingly accurate test illustrates the theoretical power of our new approach to use quantum theory as a mathematical tool to explain and predict human cognitive behaviors. We show that quantum theory, initially invented to explain order effects of measurements in physics, provides a powerful prediction for order effects of measurements in psychology.

Measurement Order Effects

One of the prime paradoxes of physics explained by quantum mechanics is that the order of measurements affects the observed statistics. For example, when testing the direction of spin ½ particles, the results depend on whether the "up-down" direction is tested before versus after the "left-right" direction (Sakurai, 1994). In the terminology of quantum theory, observables like these are defined as *incompatible*, and the theory was built on a *non-commutative* algebra of operators (Von Neumann, 1932).

Order effects of measurements are not unique to physics. It has long been recognized that the order of questions can influence human judgments and decisions (Schuman & Presser, 1981; Sudman & Bradburn, 1974). For example, the Pew Research Center conducted a telephone survey experiment during June 10-14, 2009 with a nationally representative sample of 1,502 U.S. adults. A random half of the sample was asked, "Do you approve or disapprove of the way Barack Obama is handling his job as President?" followed by "All in all, are you satisfied or dissatisfied with the way things are going in this country today?" The other half was asked the exact same questions but in the opposite order. It turns out that the presidential job approval rate was 63.38% when it was asked first and dropped to 58.58% when asked second.

Gauging public opinions is an enormously important task in any democracy. Among many challenges that survey researchers must manage, question order effects are one of the most important (Schuman & Presser, 1981; Moore, 2002). A common practice is rotating question orders between randomly-split samples to balance out question order effects. Whether the order of two questions produces significant effects can be easily tested. Denote p(AyBn) as the probability of agreeing ("yes") to question A and then disagreeing ("no") to question B, and p(BnAy) as the probability of the same answers when the questions were asked in the opposite order. Similarly, probabilities of the remaining response combinations, p(AnBy) and p(ByAn), are defined. The two order conditions produce a pair of 2×2 contingency tables, which, according to the null hypothesis, should be equivalent except for sampling error (e.g., p(AyBn) = p(BnAy)). Discrepancy from the null hypothesis is measured by χ^2. If the null hypothesis is correct, the χ^2 statistic should have a $\chi^2(3)$ distribution.

Table 1 shows χ^2 results for two Gallup survey experiments reported in a seminal article on question order effects (Moore, 2002). Each sampled around 1,000 U.S. adults using the split sample paradigm. In the first poll, people were asked whether Bill Clinton was honest and trustworthy, and whether Al Gore was honest and trustworthy. In the second poll, people were asked whether white people dislike black people, and whether black people dislike white people. Each 2×2 contingency table in Table 1 summarizes the observed proportions for the four response combinations in one question order. As shown by the χ^2 test on the order effects, both experiments produced large order effects with strikingly different patterns. Now we come backto the presidential job approval and country satisfaction questions. Is there a robust order effect for this pair of important public opinion questions? To examine this, we obtained from the Pew Research Center all its survey experiments that included this pair of questions in past 10 years. There are 26 surveys in total, with a nationally representative sample between 815 and 3,006 U.S. adults (M = 1,644, SD = 422.24). Of each sample in each survey, a random half was asked the presidential job approval question first while the other half was asked the country satisfaction question first. The χ^2 test indicates significant question order effects across the 26 surveys (see Figure 1).

Table 1: Observed proportions for each order condition, and χ^2 tests for testing order effects and the QQ equality. See Appendix on the χ^2 tests.

Observed proportions in the two different question orders

Clinton-Gore	Gy	Gn	White-Black	By	Bn
Cy	.4899	.0447	Wy	.3987	.0174
Cn	.1767	.2886	Wn	.1612	.4227
Gore-Clinton	**Cy**	**Cn**	**Black-White**	**Wy**	**Wn**
Gy	.5625	.1991	By	.4012	.0597
Gn	.0255	.2130	Bn	.1379	.4012

Discrepancy tests

	Clinton-Gore / Gore-Clinton	White-Black / Black-White
Order effects	χ^2 (3) = 10.14, p < .05	χ^2 (3) = 73.04, p < .001
QQ equality	q = -.003 χ^2 (1) = .01, p =.91	q = -.02 χ^2 (1) = .56, p = .46

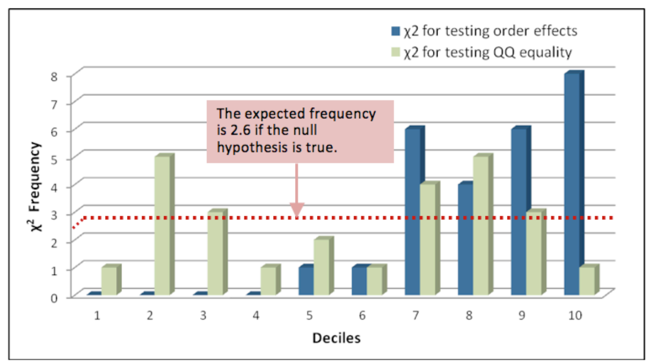

Figure 1: χ^2 frequency distributions for testing the order effects and the QQ equality. The navy bars show the observed frequencies of χ^2 values for order effects distributed across 10 categories separated at 9 deciles (.1, .2, .3, .4, .5, .6, .7, .8, .9); the green bars show those for the QQ equality test; the dotted line shows the expected frequency by the null hypothesis. The observed frequency distribution of order effects significantly differs from the expected frequency ($\chi^2(9)$ = 37.675, p < .0001), but that of the QQ equality is not ($\chi^2(9)$ = 9.5485, p = .3935). So, as predicted by the quantum model, there is a significant measurement order effects but the QQ equality holds across the 26 nationals surveys. See Appendix on the χ^2 tests.

A Quantum Model for Question Order Effects

It would be a speculative leap, however, to think that quantum theory can be applied to human behavior simply because the behavior displays measurement orders effects. Indeed, quantum models of cognition need to be rigorously tested. A precise and empirically testable prediction has been derived from a quantum model for the question order experimental paradigm (Busemeyer & Bruza, 2012; Wang & Busemeyer, in press). The model is simple, intuitive, but general. First, as illustrated in Figure 2, a person's prior belief state is represented by a unit length vector (denoted by S) within an *N*-dimensional vector space. This use of feature vectors to represent belief or knowledge is consistent with many other cognitive models of memory. Second, each answer to a question is represented by a subspace within the vector space. Each subspace corresponds to a projector (see Figure 2). Denote P_X as the projector corresponding to agreement to a question, and $I - P_X$ is the projector corresponding to disagreement to the question, where I is the identity operator. Third, how to compute response probabilities in quantum models? For example, following quantum probability rules, the probability of agreeing to question *A* and then disagreeing to question *B* equals the squared length of the result obtained by *sequentially* projecting the prior belief state on the subspace for agreeing to *A* and then on the subspace for disagreeing to *B*, that is, $p(\text{AyBn}) = \|(I-P_B)P_A \text{S}\|^2$. If the subspaces for the two questions are *incompatible* (i.e., not spanned by a common basis), then their projectors are *non-commutative* (i.e., $P_B P_A \neq P_A P_B$), and question order effects are predicted to occur.

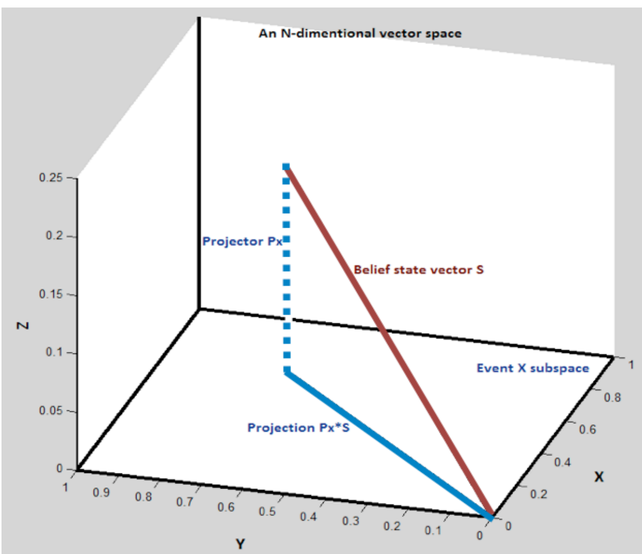

Figure 2: An illustration of basic quantum principles used in the question order model. The figure illustrates a simple 3-dimensional vector space, but the space can be arbitrarily high-dimensional. The probability of agreeing to question X is the squared length of the projection Px*S obtained by projecting the belief state S to the X-Y plane representing the subspace for agreeing to question X. If question X was asked after another question, the belief state would have already been changed by answering the preceding question, and the probability of agreeing to question X (conditioned on the preceding answer) becomes the squared length of the result obtained by projecting the adjusted belief state on the subspace for question X.

This model makes an *a priori* and precise prediction, named the Quantum Question (QQ) equality (see Appendix for proof): $[p(\text{AyBn})+p(\text{AnBy})] - [p(\text{ByAn})+p(\text{BnAy})] = 0$. Intuitively, this means, the probability of having different responses to the two questions (e.g., saying "yes" to one and "no" to the other) should remain the same across the two question orders. As shown in the proof, this equality must hold for *any* belief state and *any* pair of projectors in *any* high-dimensional vector space. This precise prediction can be easily tested empirically: if it holds, the difference in observed proportions on the left hand of the QQ equality, defined as q, should not statistically differ from zero as tested by χ^2 for difference in proportions.

The QQ equality prediction was tested using the aforementioned two Gallup data sets and was supported with high accuracy (see Table 1). To generalize the results, it was further tested using the 26 Pew national survey experiments. If it holds, the observed frequency distribution of χ^2 (shown as the green bars in Figure 1) should be distributed according to a χ^2 (1) distribution. Indeed as predicted, the observed distribution is not significantly different from the expected distribution (see Figure 1). In summary, although the 26 Pew studies exhibit significant questions order effects, there are not significant deviations from the predicted QQ equality.

Can a Classical Brain Give Rise to Quantum Cognitive Behaviors?

The surprisingly accurate predictions generated by the quantum model for question order effects is one example of an accumulating body of evidence supporting the general applicability of quantum theory for explaining a wide range of human cognitive behavior findings that are paradoxical from a classical probability perspective (Busemeyer & Bruza, 2012). This, however, leaves a question: can a classical brain give rise to behavior that follows quantum principles? Recently, mathematical physicists have provided a mathematical answer to this puzzle. Essentially, coarse measurements of a classical dynamic system typically generate *incompatible* observables that result in unresolvable uncertainty relations and entangled correlations (beim Graben & Atmanspacher, 2006; beim Graben, Filk & Atmanspacher, in press). According to quantum theory, order effects occurs for incompatible observables.

A key idea is to distinguish "ontic" states (e.g., states of a dynamic neural network) in a classical phase space from "epistemic" states (e.g., discrete choices or judgments across time) obtained from an observable. The mapping from ontic to epistemic states usually is many to one, where the epistemic states generated by an observable form a partition of the ontic phase space. Knowing the epistemic state does not completely determine the ontic state, but a sequence of measurements across time refines the partition of the phase space. In the limit, the partition reaches a "finest dynamic refinement," denoted by $\wp$. Now suppose two observables (***f,g***) produce different finest dynamic refinements ($\wp_f \neq \wp_g$) and neither converge to the identity partition, as illustrated in Figure 3. This means that no ontic state is accessible by a sequence of measurements from either observable. Then there exists an epistemic state (a set of ontic states) $F_a \in \wp_{\boldsymbol{f}}$ that determines the value *a* produced by the observable ***f***, but the value of the observable ***g*** must remain dispersive. Likewise, there exists an epistemic state

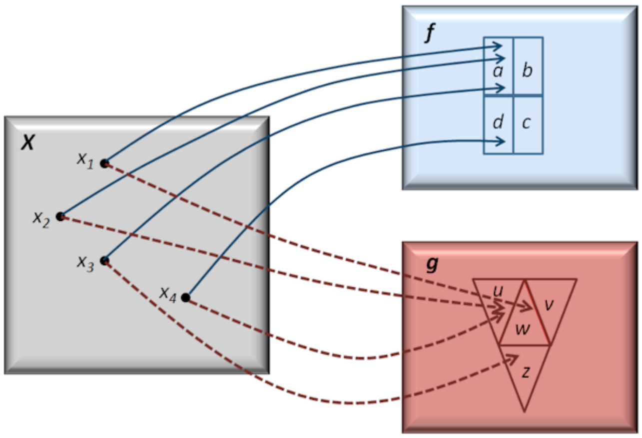

Figure 3. An illustration showing how uncertainty relations are generated by coarse descriptions of classical dynamic systems. The underlying classical phase space *X* is inconsistently partitioned by two different observables, *f* and *g*. The cell within *X* that always assigns a value *a* to the observable *f* assigns a range of different possible values (*w,u,z*) to the observable *g*. In this case, there exists an epistemic state that determines the value *a* produced by the observable *f*, but the value of the observable *g* must remain dispersive.

(a set of ontic states) $G_v \in \wp_{\boldsymbol{g}}$ that determines the value *v* produced by the observable ***g***, but the value of the observable ***f*** remains dispersive. It is impossible to simultaneously determine the value *a* from observable ***f*** and the value *v* from the observable ***g*** with arbitrary precision, so that the two observables are incompatible. Consequently, the partitions generated by the two incompatible observables produce incompatible Boolean algebras of events, and the entire collection forms a partial rather than a complete Boolean algebra. Quantum theory is specifically suitable to assign probabilities to events defined on a partial Boolean algebra.

Discussion

Scientists are still far from understanding how mental states emerge from the neural substrates. It is too early to conclude whether or not quantum physics plays a significant role in neural processing. Nevertheless, even if the brain is classical, the ubiquitous nature of incompatible observables provides a good reason to consider using quantum theory as a mathematical tool for predicting human behavior (Busemeyer & Bruza, 2012; Khrennikov, 2010). As our quantum question order model encapsulates and illustrates, at least four motivations drives the development of this new field of quantum cognition. (a) Judgments and decisions are not simply read out from memory, but rather, they are constructed from the cognitive state for the question at hand; and (b) drawing a conclusion from one judgment or decision changes the context and disturbs the cognitive system, which then (c) affects the next judgment or decision, producing order effects, so that (d) human judgments and decisions do not obey the commutative rule of Boolean logic. If we replace "judgments or decisions" with "physical measurements" and replace "cognitive system" with "physical system," then these are exactly the same reasons that forced physicists to develop quantum theory in the first place. Traditionally, quantum theory has rarely been applied outside of physics, but now a growing number of researchers are successfully using it to explain human cognitive behavior.

Acknowledgments

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Appendix

1. Proof of the QQ equality.

Here we briefly introduce the basic axioms of quantum theory and then derive the QQ equality. We use the Dirac bracket notation so that $\langle S|T\rangle$ represents the inner product between two vectors. According to quantum theory, events represented as subspaces of a Hilbert space. Corresponding to each event A there is a orthogonal projector $\boldsymbol{P}_A$. The state of a quantum system is represented by a unit length vector S within the Hilbert space. The probability of event A equals the squared length of the projection $p(A) = ||\boldsymbol{P}_A S||^2$. If event A is observed, then the state is updated according to Lüder's rule $S_A = \boldsymbol{P}_A S / ||\boldsymbol{P}_A S||$.

Define S as the initial state. Denote the projector for saying yes to question C as $\boldsymbol{P_C}$ and denote $\boldsymbol{P}_G$ as the projector for saying yes to question G. We start by expanding the probability for answering "yes" to question C:

$$||\boldsymbol{P_C} \cdot S||^2 = ||\boldsymbol{P_C} \cdot \boldsymbol{I} \cdot S||^2 = ||\boldsymbol{P_C} \cdot (\boldsymbol{P_G} + (\boldsymbol{I}-\boldsymbol{P_G})) \cdot S||^2 = ||\boldsymbol{P_C} \cdot \boldsymbol{P_G} \cdot S + \boldsymbol{P_C} \cdot (\boldsymbol{I}-\boldsymbol{P_G}) \cdot S||^2$$
$$= ||\boldsymbol{P_C} \cdot \boldsymbol{P_G} \cdot S||^2 + ||\boldsymbol{P_C} \cdot (\boldsymbol{I}-\boldsymbol{P_G}) \cdot S||^2 + \langle S| \cdot \boldsymbol{P_G} \cdot \boldsymbol{P_C} \cdot \boldsymbol{P_C} \cdot (\boldsymbol{I}-\boldsymbol{P_G})|S\rangle + \langle S|(\boldsymbol{I}-\boldsymbol{P_G}) \cdot \boldsymbol{P_C} \cdot \boldsymbol{P_C} \cdot \boldsymbol{P_G}|S\rangle$$
$$= ||\boldsymbol{P_C} \cdot \boldsymbol{P_G} \cdot S||^2 + ||\boldsymbol{P_C} \cdot (\boldsymbol{I}-\boldsymbol{P_G}) \cdot S||^2 + \langle S|\boldsymbol{P_G} \cdot \boldsymbol{P_C} \cdot \boldsymbol{P_C} \cdot (\boldsymbol{I}-\boldsymbol{P_G})|S\rangle + \langle S| \cdot \boldsymbol{P_G} \cdot \boldsymbol{P_C} \cdot \boldsymbol{P_C} \cdot (\boldsymbol{I}-\boldsymbol{P_G})|S\rangle^*$$
$$= ||\boldsymbol{P_C} \cdot \boldsymbol{P_G} \cdot S||^2 + ||\boldsymbol{P_C} \cdot (\boldsymbol{I}-\boldsymbol{P_G}) \cdot S||^2 + 2 \cdot \mathrm{Re}[\langle S|\boldsymbol{P_G} \cdot \boldsymbol{P_C} \cdot \boldsymbol{P_C} \cdot (\boldsymbol{I}-\boldsymbol{P_G})|S\rangle]$$
$$= ||\boldsymbol{P_C} \cdot \boldsymbol{P_G} \cdot S||^2 + ||\boldsymbol{P_C} \cdot (\boldsymbol{I}-\boldsymbol{P_G}) \cdot S||^2 + 2 \cdot \mathrm{Re}[\langle S|\boldsymbol{P_G} \cdot \boldsymbol{P_C} \cdot (\boldsymbol{I}-\boldsymbol{P_G})|S\rangle],$$

and the latter follows from the idempotent property of projectors. (The symbol x^* used in the above derivation refers to the complex conjugate of x.) Define the total probability to say yes to question C when G was asked first as

$$TP_C = ||\boldsymbol{P_G} \cdot S||^2 \cdot ||\boldsymbol{P_C} \cdot S_G||^2 + ||\boldsymbol{P_{\sim G}} \cdot S|^2 \cdot ||\boldsymbol{P_C} \cdot S_{\sim G}||^2$$
$$= ||\boldsymbol{P_C} \cdot \boldsymbol{P_G} \cdot S||^2 + ||\boldsymbol{P_C} \cdot (\boldsymbol{I}-\boldsymbol{P_G}) \cdot S||^2.$$

An order effect for question C when G was asked first expressed as

$$C_C = TP_C - ||\boldsymbol{P_C} \cdot S||^2 = -2 \cdot \mathrm{Re}[\langle S|\boldsymbol{P_G} \cdot \boldsymbol{P_C} \cdot (\boldsymbol{I}-\boldsymbol{P_G})|S\rangle].$$

Immediately we see that if $\boldsymbol{P_G}$ and $\boldsymbol{P_C}$ commute so that $\boldsymbol{P_G} \cdot \boldsymbol{P_C} = \boldsymbol{P_C} \cdot \boldsymbol{P_G}$ then $\boldsymbol{P_G} \cdot \boldsymbol{P_C} \cdot (\boldsymbol{I}-\boldsymbol{P_G}) = \boldsymbol{P_C} \cdot \boldsymbol{P_G} \cdot (\boldsymbol{I}-\boldsymbol{P_G}) = 0$ and we predict NO order effect. Thus non-commuting projectors are a necessary condition for order effects. Now let us re-examine

$$C_C = TP_C - ||\boldsymbol{P_C} \cdot S||^2 = -2 \cdot \mathrm{Re}[\langle S|\boldsymbol{P_G} \cdot \boldsymbol{P_C} \cdot (\boldsymbol{I}-\boldsymbol{P_G})|S\rangle].$$
$$= -2 \cdot \mathrm{Re}[\langle S|\boldsymbol{P_G} \cdot \boldsymbol{P_C}|S\rangle - \langle S|\boldsymbol{P_G} \cdot \boldsymbol{P_C} \cdot \boldsymbol{P_G}|S\rangle]$$
$$= -2 \cdot \mathrm{Re}[\langle S|\boldsymbol{P_G} \cdot \boldsymbol{P_C}|S\rangle - ||\boldsymbol{P_C} \cdot \boldsymbol{P_G} \cdot S||^2]$$
$$= -2 \cdot \mathrm{Re}[\langle S|\boldsymbol{P_G} \cdot \boldsymbol{P_C}|S\rangle] + 2 \cdot ||\boldsymbol{P_C} \cdot \boldsymbol{P_G} \cdot S||^2$$
$$= 2 \cdot ||\boldsymbol{P_G} \cdot S||^2 \cdot ||\boldsymbol{P_C} \cdot S_G||^2 - 2 \cdot \mathrm{Re}[\langle S|\boldsymbol{P_G} \cdot \boldsymbol{P_C}|S\rangle] \ .$$

In general, the inner product is a complex number which always can be expressed as $\langle S|\boldsymbol{P_G} \cdot \boldsymbol{P_C}|S\rangle = |\langle S|\boldsymbol{P_G} \cdot \boldsymbol{P_C}|S\rangle| \cdot [\cos(\phi) + i \cdot \sin\phi]$. The real part equals $\mathrm{Re}[\langle S|\boldsymbol{P_G} \cdot \boldsymbol{P_C}|S\rangle] = |\langle S|\boldsymbol{P_G} \cdot \boldsymbol{P_C}|S\rangle| \cdot \cos(\phi)$. By defining the ratio

$$R = |\langle S|\boldsymbol{P_G} \cdot \boldsymbol{P_C}|S\rangle| \ / \ (||\boldsymbol{P_C} \cdot S|| \cdot ||\boldsymbol{P_G} \cdot S||),$$

then according to the Cauchy-Schwarz inequality, $0 \leq R \leq 1$. Finally we can express

$$C_C = TP_C - ||\boldsymbol{P_C} \cdot S||^2 = 2 \cdot ||\boldsymbol{P_C} \cdot \boldsymbol{P_G} \cdot S||^2 - 2 \cdot R \cdot \cos(\phi) \cdot ||\boldsymbol{P_C} \cdot S|| \cdot ||\boldsymbol{P_G} \cdot S||$$
$$= 2 \cdot ||\boldsymbol{P_G} \cdot S||^2 \cdot ||\boldsymbol{P_C} \cdot S_G||^2 - 2 \cdot \theta \cdot ||\boldsymbol{P_C} \cdot S|| \cdot ||\boldsymbol{P_G} \cdot S||,$$

with $\theta = R \cdot \cos(\phi)$ and $-1 \leq \theta \leq +1$, which is the similarity index referred to in the main text. Similarly, the order effect for question G when C was asked first equals

$$C_G = TP_G - ||\boldsymbol{P_G} \cdot S||^2 = 2 \cdot ||\boldsymbol{P_G} \cdot \boldsymbol{P_C} \cdot S||^2 - 2 \cdot \mathrm{Re}[\langle S|\boldsymbol{P_C} \cdot \boldsymbol{P_G}|S\rangle], \text{ but } \mathrm{Re}[\langle S|\boldsymbol{P_C} \cdot \boldsymbol{P_G}|S\rangle] = \mathrm{Re}[\langle S|\boldsymbol{P_G} \cdot \boldsymbol{P_C}|S\rangle]$$

so that

$$C_G = 2 \cdot ||\boldsymbol{P_C} \cdot S||^2 \cdot ||\boldsymbol{P_G} \cdot S_C||^2 - 2 \cdot \theta \cdot ||\boldsymbol{P_C} \cdot S|| \cdot ||\boldsymbol{P_G} \cdot S||.$$

These two order effects share the same term, $2 \cdot \theta \cdot ||\boldsymbol{P_C} \cdot S|| \cdot ||\boldsymbol{P_G} \cdot S||$, and therefore together they imply the relation

$$0 = (2 \cdot ||\boldsymbol{P_C P_G} \cdot S||^2 - C_C) - (2 \cdot ||\boldsymbol{P_G P_C} \cdot S||^2 - C_G)$$
$$= (2 \cdot ||\boldsymbol{P_C P_G} \cdot S||^2 - ||\boldsymbol{P_C P_G} \cdot S||^2 - ||\boldsymbol{P_C P_{\sim G}} \cdot S||^2 + ||\boldsymbol{P_C} \cdot S||^2) - (2 \cdot ||\boldsymbol{P_G P_C} \cdot S||^2 - ||\boldsymbol{P_G P_C} \cdot S||^2 - ||\boldsymbol{P_G P_{\sim C}} \cdot S||^2 + ||\boldsymbol{P_G} \cdot S||^2)$$

$= (\|\boldsymbol{P_C P_G} \cdot S\|^2 - \|\boldsymbol{P_C P_{\sim G}} \cdot S\|^2 + \|\boldsymbol{P_C} \cdot S\|^2) - (\|\boldsymbol{P_G P_C} \cdot S\|^2 - \|\boldsymbol{P_G P_{\sim C}} \cdot S\|^2 + \|\boldsymbol{P_G} \cdot S\|^2)$

$= [\|\boldsymbol{P_C P_G} \cdot S\|^2 - \|\boldsymbol{P_C P_{\sim G}} \cdot S\|^2 + (\|\boldsymbol{P}_G \cdot S_C\|^2 + \|\boldsymbol{P}_{\sim G} \cdot S_C\|^2) \cdot \|\boldsymbol{P_C} \cdot S\|^2] - [\|\boldsymbol{P_G P_C} \cdot S\|^2 - \|\boldsymbol{P_G P_{\sim C}} \cdot S\|^2 + (\|\boldsymbol{P}_C \cdot S_G\|^2 + \|\boldsymbol{P}_{\sim C} \cdot S_G\|^2) \cdot \|\boldsymbol{P_G} \cdot S\|^2]$

$= (\|\boldsymbol{P_C P_G} \cdot S\|^2 - \|\boldsymbol{P_C P_{\sim G}} \cdot S\|^2 + \|\boldsymbol{P}_G \cdot S_C\|^2 \|\boldsymbol{P_C} \cdot S\|^2 + \|\boldsymbol{P}_{\sim G} \cdot S_C\|^2 \|\boldsymbol{P_C} \cdot S\|^2) - (\|\boldsymbol{P_G P_C} \cdot S\|^2 - \|\boldsymbol{P_G P_{\sim C}} \cdot S\|^2 + \|\boldsymbol{P}_C \cdot S_G\|^2 \|\boldsymbol{P_G} \cdot S\|^2 + \|\boldsymbol{P}_{\sim C} \cdot S_G\|^2 \|\boldsymbol{P_G} \cdot S\|^2)$

$= (\|\boldsymbol{P_C P_G} \cdot S\|^2 - \|\boldsymbol{P_C P_{\sim G}} \cdot S\|^2 + \|\boldsymbol{P}_G \boldsymbol{P_C} \cdot S\|^2 + \|\boldsymbol{P}_{\sim G} \boldsymbol{P_C} \cdot S\|^2) - (\|\boldsymbol{P_G P_C} \cdot S\|^2 - \|\boldsymbol{P_G P_{\sim C}} \cdot S\|^2 + \|\boldsymbol{P}_C \boldsymbol{P_G} \cdot S\|^2 + \|\boldsymbol{P}_{\sim C} \boldsymbol{P_G} \cdot S\|^2)$

$= (\|\boldsymbol{P}_{\sim G} \boldsymbol{P_C} \cdot S\|^2 - \|\boldsymbol{P_C P_{\sim G}} \cdot S\|^2) - (\|\boldsymbol{P}_{\sim C} \boldsymbol{P_G} \cdot S\|^2 - \|\boldsymbol{P_G P_{\sim C}} \cdot S\|^2)$

$= (\|\boldsymbol{P}_{\sim G} \boldsymbol{P_C} \cdot S\|^2 + \|\boldsymbol{P_G P_{\sim C}} \cdot S\|^2) - (\|\boldsymbol{P}_{\sim C} \boldsymbol{P_G} \cdot S\|^2 + \|\boldsymbol{P_C P_{\sim G}} \cdot S\|^2) = 0.$ Q.E.D.

The last line is the QQ equality expressed as quantum probabilities.

2. χ^2 tests used in Table 1.

First we present the χ^2 test for order effects. Define n_{YN} as the frequency of saying "yes" to question C when C was asked first and saying "no" to question G when G was asked second, and the other combinations of answers are defined similarly. Define $n = n_{YY} + n_{YN} + n_{NY} + n_{NN}$. Define m_{YN} as the frequency of "yes" to question G when G was asked first and "no" to question C when C was asked second, and the other combinations of answers are defined similarly. Define $m = m_{YY} + m_{YN} + m_{NY} + m_{NN}$. The log likelihood for the unconstrained model that allows order effects is defined by

$$G_U = [n_{YY} \cdot ln(n_{YY}/n) + n_{YN} \cdot ln(n_{YN}/n) + n_{NY} \cdot ln(n_{NY}/n) + n_{NN} \cdot ln(n_{NN}/n) + m_{YY} \cdot ln(m_{YY}/m) + m_{YN} \cdot ln(m_{YN}/m) + m_{NY} \cdot ln(m_{NY}/m) + m_{NN} \cdot ln(m_{NN}/m)]. \quad (1a)$$

The log likelihood for the constrained model that assumes no order effects is defined by

$$G_C = [(n_{YY} + m_{YY}) \cdot ln((n_{YY} + m_{YY})/(n+m)) + (n_{YN} + m_{NY}) \cdot ln((n_{YN} + m_{NY})/(n+m)) + (n_{NY} + m_{YN}) \cdot ln((n_{NY} + m_{YN})/(n+m)) + (n_{NN} + m_{NN}) \cdot ln((n_{NN} + m_{NN})/(n+m))]. \quad (1b)$$

The χ^2 statistic is defined by the difference $\chi^2 = -2 \cdot (G_C - G_U)$. The unconstrained model involves $(4-1) + (4-1) = 6$ free parameters and the constrained model involves $4-1 = 3$ free parameters, and so the χ^2 statistic has $df = 3$.

Next we define the χ^2 test for the QQ equality. The log likelihood for the unconstrained model is defined as

$$G_U = [(n_{YN} + n_{NY}) \cdot ln((n_{YN} + n_{NY})/n) + (n_{YY} + n_{NN}) \cdot ln((n_{YY} + n_{NN})/n) = (m_{YN} + m_{NY}) \cdot ln((m_{YN} + m_{NY})/m) + (m_{YY} + m_{NN}) \cdot ln((m_{YY} + m_{NN})/m)]. \quad (2a)$$

The log likelihood for the model constrained by the QQ equality equals

$$G_C = [(n_{YN} + n_{NY} + m_{YN} + m_{NY}) \cdot ln((n_{YN} + n_{NY} + m_{YN} + m_{NY})/(n + m)) + (n_{YY} + n_{NN} + m_{YY} + m_{NN}) \cdot ln((n_{YY} + n_{NN} + m_{YM} + m_{NN})/(n + m))]. \quad (2b)$$

The χ^2 statistic is defined by the difference $\chi^2 = -2 \cdot (G_C - G_U)$. The unconstrained model involves $(2-1) + (2-1) = 2$ free parameters and the constrained model involves $2-1 = 1$ free parameter, and so the χ^2 statistic has $df = 1$.

3. χ^2 tests used in Figure 1.

First we describe the χ^2 test for order effects. The χ^2 statistic for testing an order effect for each of the 26 data sets was computed using Equations 1a and 1b defined above, producing 26 observed χ^2 values. If the null hypothesis is correct, these should be distributed according to a χ^2 distribution with $df = 6$. Ten categories were constructed by computing the 9 category bounds: .5844 equals the 10th percentile, 1.0052 equals the 20th percentile, 1.4237 equals the 30th percentile, 1.8692 equals the 40th percentile, 2.3660 equals the 50th percentile, 3.9462 equals the 60th percentile, 3.6649 equals the 70th percentile, 4.6416 equals the 80th percentile, and 6.2514 equals the 90th percentile. (For example, $Pr[\chi^2(6) < 6.2514 \mid H_0] = .90$.) These category bounds divide the expected frequency distribution (under the null hypotheses) into two 10 equally likely categories, with 2.6 expected frequency within each of the 10 categories using these cutoffs. Then frequency of the 26 observed χ^2 values were counted for each category. Denote f_i as the observed frequency for category $i = 1,10$. The log likelihood for the unconstrained model equals

$$G_U = \textstyle\sum_i f_i \cdot ln(f_i /26). \quad (3a)$$

The log likelihood for the expected frequencies according to the null hypothesis equals

$$G_C = \textstyle\sum_i f_i \cdot ln(2.6 /26) . \quad (3b)$$

The χ^2 statistic is defined by the difference $\chi^2 = -2 \cdot (G_C - G_U)$. The unconstrained model involves $10-1 = 9$ free parameters and the constrained model has no free parameters, and so the χ^2 statistic has $df = 9$.

Next we describe the χ^2 test for the QQ equality. The χ^2 statistic for testing the QQ equality for each of the 26 data sets was computed using Equations 2a and 2b defined above, producing 26 observed χ^2 values. If the null hypothesis is correct, these should be distributed according to a χ^2 distribution with $df = 1$. Ten categories were constructed by computing the 9 category bounds: .0158 equals the 10th percentile, .0642 equals the 20th percentile, .1485 equals the 30th percentile, .2750 equals the 40th percentile, .4549 equals the 50th percentile, .7083 equals the 60th percentile, 1.0742 equals the 70th percentile, 1.6424 equals the 80th percentile, and 2.7055 equals the 90th percentile. (For example, $Pr[\chi^2(1) < 2.7055 \mid H_0] = .90$.) These category bounds divide the expected frequency distribution (under the null hypotheses) into two 10 equally likely categories, with 2.6 expected frequency within each of the 10 categories using these cutoffs. Then frequency of the 26 observed χ^2 values were counted for each category. Denote f_i as the observed frequency for category $i = 1,10$. Then Equations 3a and 3b were used to compute the log likelihoods of the unconstrained and constrained models. Once again, the χ^2 statistic is defined by the difference $\chi^2 = -2 \cdot (G_C - G_U)$. The unconstrained model involves $10-1 = 9$ free parameters and the constrained model has no free parameters, and so the χ^2 statistic has $df = 9$.