A Dynamical Model of Risky Choice

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Abstract

Individuals make decisions under uncertainty every day based on incomplete information concerning the potential outcome of the choice or chance levels. The choices individuals make often deviate from the rational or mathematically objective solution. Accordingly, the dynamics of human decision-making are difficult to capture using conventional, linear mathematical models. Here, we present data from a two-choice task with variable risk between sure loss and risky loss to illustrate how a simple nonlinear dynamical system can be employed to capture the dynamics of human decision-making under uncertainty (i.e., multi-stability, bifurcations). We test the feasibility of this model quantitatively and demonstrate how the model can account for up to 86% of the observed choice behavior. The implications of using dynamical models for explaining the nonlinear complexities of human decision-making are discussed, as well as the degree to which nonlinear dynamical systems theory might offer an alternative framework for understanding human decision-making processes.

Keywords: Decision-making; Complex Systems; Dynamical Systems Modeling; Risky Choice; Multi-stability; Phase Transitions.

Introduction

Decision-making is part of almost everything humans do. Decisions can be commonplace or trivial but can also have lifelong consequences. Therefore, it is important to understand how individuals make decisions and how various factors play a role in decision-making processes. One such factor is uncertainty, which occurs in situations where there is limited information, ambiguous information, or unreliable information. Another factor is risk, which is different from uncertainty and can be defined as 'probabilized' uncertainty (Etner, Jeleva, & Tallon, 2010).

Johnson and Busemeyer (2010) distinguish three major streams of development in decision theory: normative research, descriptive research, and the computational approach. While the normative approach defines what would be the optimal decision in a given situation, descriptive research describes how humans actually decide. For example, this approach has lead to the insight that individuals are sensitive to framing. When a decision is framed in terms of potential loss, the majority of participants avoid taking risk, but when the same decision is framed in terms of potential gain, the majority of participants do take risk (Tversky & Kahneman, 1974). In another study, Kahneman and Tversky (1979; 1983) showed that risks with low probabilities are either grossly overweighed, or completely neglected, and that there is large heterogeneity among individuals. Specifically, individuals show more variability in deciding about potential loss than potential gain (Tversky & Kahneman, 1981). These examples suggest that human decision-making behavior under uncertainty can well be described using a nonlinear, dynamic narrative; individual decision behavior is highly context-specific, unstable, and heterogeneous.

The aim of this article is therefore to investigate the feasibility of extending current efforts in decision science towards a nonlinear, dynamical approach.

Decision-Making and Multi-Stability

Heterogeneity, multi-stability, and context-sensitivity in general, are all strong indications that decision-making under uncertainty is characterized by nonlinear dynamics. A multi-stable system can, for the same input, settle in more than one possible internal stable state. A possible consequence of multi-stability is *hysteresis*, which is the phenomenon whereby a system's immediate history influences the current state of the system. Sir James Alfred Ewing first coined the term hysteresis while observing the phenomenon in magnetic materials (Ewing, 1881).

Figure 1A displays hysteresis in the magnetization and demagnetization of a magnet as a result of varying strength of the magnetic force. Depending on the direction of change of the magnetic field, the change from magnetization in one direction to the opposite direction occurs at a different moment. The system has a primitive form of memory, and remains in an existing stable state longer than expected. The opposite of hysteresis, *reversed hysteresis*, can also occur in multi-stable systems. Rather than remaining in the existing stable state longer (as with hysteresis), the system changes to another stable state sooner.

Hysteresis and reversed hysteresis are important indications of nonlinearity (Kelso, 1995). Hysteresis in behavioral dynamics has been found in body-scaled transitions like grasping of objects (Richardson, Marsh, & Baron, 2007; Lopresti-Goodman, Turvey, & Frank, 2011), speech categorization (Tuller, Case, Ding, & Kelso, 1994), perception of whether a slanted surface supports upright standing (Fitzpatrick, Carello, Schmidt, & Corey, 1994), and problem-solving (Stephen, Boncoddo, Magnuson, & Dixon, 2009).

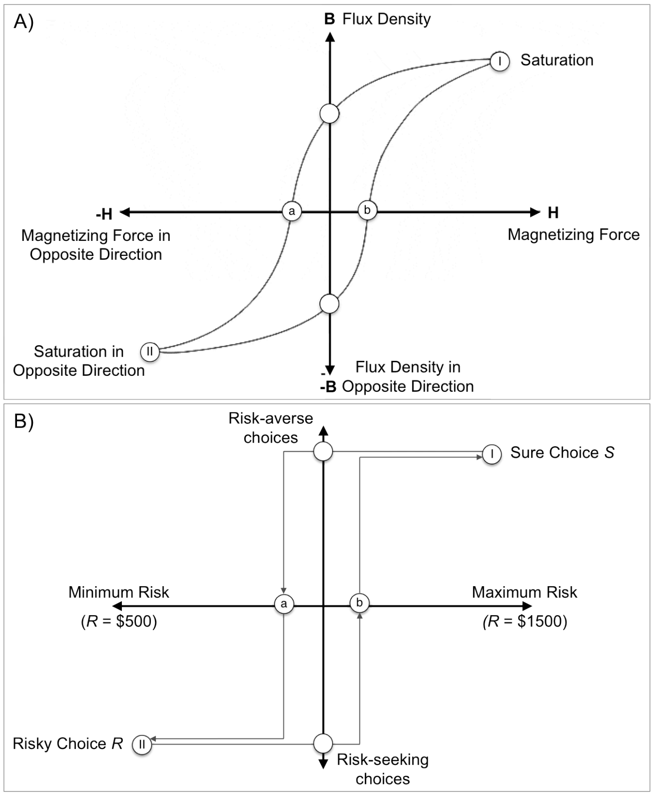

Figure 1: Hysteresis in magnets (A) and risky choice (B). A) A magnet is magnetized by a magnetizing force *H*, into direction *B* (state I). If the strength of *H* is then slowly decreased, the saturation of the magnet will change until it becomes fully magnetized into the opposite direction *–B* (state II). If *H* is increased again, the change towards saturation in the positive direction *B* happens at a different value for the strength of the magnetic force *H*. B) See text.

In order to test for hysteresis and reversed hysteresis in decision-making, we will adopt a standard model of risky decision behavior with the implicit assumption that real-world decisions under uncertainty have the same properties as a monetary gamble (Hertwig & Erev, 2009). Figure 2 displays a typical example of the type of monetary gamble researchers use to study risky decision behavior; the choice between a sure option A, and a risky option B (Kahneman et al., 1981). Choice A and B have the same expected values, thus from a rational choice perspective, they are equivalent.

Choose between:

A. a sure loss of $750
B. 75% chance to lose $1000, and 25% change to lose nothing

Figure 2: Example of a risky choice.

This kind of gamble, hereafter called *risky choice*, can be formulated in terms of potential loss (as in the example above) or in terms of potential gain. For the remainder of this article, we will focus on loss, as potential loss is expected to maximize the variability among participants. The parameters in a risky choice are the probability to lose *P*, and the values of *R* and *S*. The outcome is either a *risk-seeking* choice for *R* or a *risk-avoiding* choice for *S*.

Finding hysteresis or reversed hysteresis in risky choice behavior will provide evidence that decision-making under uncertainty is indeed characterized by nonlinear dynamics.

Sequential Risky Choice

Two key components to finding hysteresis or reversed hysteresis in risky choice are to (1) change the context in two opposite directions, and (2) do this in a systematic way. It is necessary to find an input parameter for which, at different values, the system's output can have opposite, or at least, qualitatively different values. In risky choice, the key parameter that drives the choice between risk-seeking and risk-avoiding behavior is the amount of risk that is present in *R*. There are several ways to vary the amount of risk in *R*; we have opted to manipulate the value of the risky loss (in $, a high value of *R* corresponds with a high risk). Only when the value of *R* is first increased and then decreased or vice versa, there will be an opportunity to observe hysteresis and/or reversed hysteresis. A *sequential* risky choice task is therefore a sequence of consecutive risky choices between *S* and *R*[1], in which the value of *R* is either increased or decreased in a step-wise fashion.

In a sequential choice task, hysteresis looks like this: A decision-maker is presented with a risky choice where the risk in *R* is minimal (relative to *S*), and chooses *R*. Next, the decision-maker is presented with a second risky choice, in which the risk in *R* is slightly higher. Next, another risky choice occurs that is even riskier, and so on. All the while the decision-maker continues choosing *R*. Then, at some *switch-point* (see definition below), when the risk in *R* has become too high, the decision-maker will switch to choosing *S* and continue to do so until the risk in *R* is maximal (relative to *S*). Then, the whole process is reversed by decreasing the risk in *R* again, causing the decision-maker to switch back from choosing *S* to choosing *R* at another switch-point. If the second switch occurs for a lower risk in *R* than the first, we have found an indication of hysteresis. If the second switch occurs for a higher risk in *R* than the first, we have found an indication of reversed hysteresis (see also figure 1B).

Method

Participants and Design Thirty-six undergraduate students from the University of Cincinnati were presented with three

1 Note that objectively, in each risky choice, *S* is the better choice as soon as the sure loss of *S* is lower than the expected value of *R*, while *R* is the better choice as soon as the expected value of *R* becomes lower than the sure loss of *S*.

sets of sequential risky choices between a risky loss *R* and a sure loss *S*. In the first and third set, the amount of risk in *R* was systematically varied, either in increasing, and then decreasing order (*ID*), or vice versa (*DI*). The second set contained the same choices in randomized order to mediate carry-over effects between the first and third sets. Half of the students were presented first with the *ID* set, followed by the random set and the *DI* set. The other half started with the *DI* set. The value of *R* ranged from \$1500 to \$525, with increments of \$25. The probability to lose this amount P = 75%, and S = \$750. The total amount of choices was 238. After completion of the sequential risky choice task, the students participated in a short money-free version of the balloon analogue risk task (BART), (Lejuez et al., 2002).

Stimulus/Apparatus All stimuli were variations of the example in Figure 2, and contained the values for *P*, *R*, and *S*. In total, 40 different values of *R* (ranging from \$525 to \$1500 with increments of \$25) were presented either on the left side of the screen, with the value of *S* on the right, or vice versa. The stimuli were presented on an iMac, and a cordless computer mouse (Apple Inc.[2]) was used to select the choices, both were run using PsychToolbox software (Brainard, 1997). The BART stimuli were presented on a different computer monitor (Dell™) and responses made using a standard computer mouse (Logitech™) were recorded using BART software made available online.

Procedure Participants provided their written consent and received instructions about the sequential risky choice task. Participants were seated in front of the computer screen that displayed the various choices and were instructed to indicate their choice preferences using the mouse. After completion of the sequential choice task, participants received instruction about the BART. They again sat in front of a computer screen on which the stimuli were displayed and were instructed to respond using the mouse.

Results

Choice outcomes of one-fourth (22%) of the participants showed no change at all. This is consistent with an earlier experiment with a smaller range of risk in *R* (from \$725-\$1175), in which 27% of the participants showed no change.

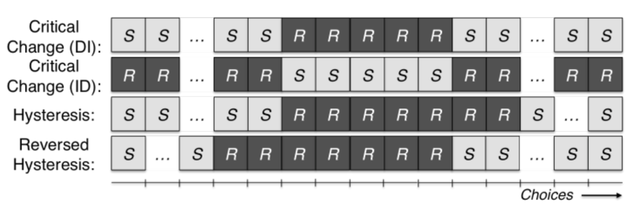

Figure 3: Model changes between choices for *R* and *S*. *Critical change* is defined as the situation where a participant switches from *S* (*R*) to *R* (*S*) for the same amount of risk in the first and second half of an *ID* or *DI* sequence. *Hysteresis* is defined as the situation where a participant switches from *S* (*R*) to *R* (*S*) later in the second half on an *ID* or *DI* sequence. *Reversed hysteresis* is defined as the situation where a participant switches from *S* (*R*) to *R* (*S*) earlier in the first half on an *ID* or *DI* sequence.

The remaining 28 participants switched between risk-seeking and risk-averse choices at least once per sequence (M = 3.8 fluctuations[3], SD = 3.4). Using an automated search algorithm, two switch-points[4] per *ID* and *DI* sequence were determined for each participant. Based on the locations of the switch-points, most participants (48%) showed critical change, followed by reversed hysteresis (39%), and hysteresis (13%), see Figure 3 for details. The average value of the risk in *R* for switches from *R* to *S* was \$1000 (SD = \$215), and from *S* to *R*, \$941 (SD = \$174) indicating that overall, participants were risk-averse ($p <$ 0.0001). The distance between the two switch-points for the *DI* and *ID* sequences was significantly larger compared to the random sequences $t(27) = 3.61$, $p = .001$, $d = .95$.

Switching under time-constraint

22-27% of participants in a sequential risky choice task do not show any change at all. A closer look revealed that all of these participants were presented with the *DI* sequence first, and consistently chose *R*. One explanation could be that for about one-fourth of participants, the attractor for *S* is non-existent. Another explanation is that the initial conditions strengthen the attractor for *R* relative to *S* such that the changing constraints provide too little perturbation to the system. A small follow-up study (N = 16) was therefore conducted with the only difference being that participants were instructed to decide as quickly as possible while still using the available information on the screen. It was hypothesized that this speed manipulation would destabilize the initial strength of the attractor for *R*. All 16 participants switched at least once between *S* and *R* (M = 10.8 fluctuations, SD = 12.5), and the relation between the speed manipulation and the absence of 'no change' participants is significant, $\chi(1, N = 52) = 4.20$, $p = .04$. The speed manipulation increased variability and caused participants to be more sensitive to changing risk constraints. This is consistent with observations that time pressure influences decision-makers' strategy selection (see Edland & Svenson, 1993 for a review).

2 This is an independent publication and has not been authorized, sponsored, or otherwise approved by Apple Inc.

3 A *fluctuation* is defined as each choice that is different from the previous choice.

4 A *switch-point* is defined as the closest fluctuation to the middle choice for which; in case of an *ID* sequence, the number of *R* choices in between this fluctuation and the first *S* choice in a continuous stretch of *S* choices spanning the middle, is less than the number of *S* in between. In case of a *DI* sequence, it is the other way around.

Varying increments of R

Increasing the value of *R* in increments of \$25 results in a high predictability of the choices in the *DI* and *ID* sequences. This could have mediated the amount of reversed hysteresis in our sample. A follow-up study was therefore conducted in which the increments were sampled from an *N*(25,1), *N*(25,2), *N*(25,4), *N*(25,8), and *N*(25,26) distribution respectively. The maximum and minimum values of *R* (\$525 and \$1500) were maintained. Figure 4 shows the distribution of types of choice behavior for the fixed increments (N = 36), and varying increments (N = 50; 10 each).

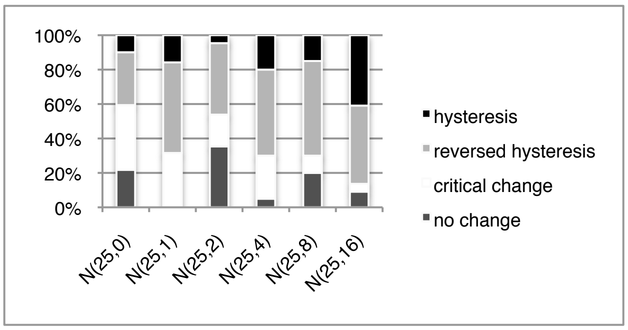

Figure 4: Distribution of types of choice behavior for varying increments of the value of *R*.

There is a main effect of sequence type (*ID* or *DI*; $p < .001$), and order (*DI* or *ID* first, p < .001) on the difference between the two switch-points, but not of the amount of variability. However, the distribution of the four types of change behavior did differ by the amount of variation in the increments of *R*, $\chi^2(12, N = 171) = 28.09$, $p < .01$, with a positive trend for the amount of participants that showed hysteresis and reversed hysteresis.

Nonlinear Dynamical Modeling

Multi-stability in switching behavior is problematic for most linear models but can be accounted for by a *nonlinear dynamical system* (e.g. Cho, Jones, Braver, Holmes, & Cohen, 2002; Roxin & Ledberg, 2008). A dynamical system is a mathematical concept where the time dependence of a state variable (a variable that describes a certain quantity of a system that we are interested in, like position or concentration) is described using a fixed rule. In a *nonlinear* dynamical system, this fixed rule is nonlinear, and the system therefore does not satisfy the additivity and homogeneity properties that are necessary for linearity.

Examples of applications of (nonlinear) dynamical modeling to human behavior are vision (for example Fürstenau, 2006), speech (Kelso, Saltzman, & Tuller, 1986; Tuller et al., 1994), language (for example Spivey, Grosjean, & Knoblich, 2005), motor and neural dynamics (Haken, Kelso, & Bunz, 1985; Schöner & Kelso, 1988, Kelso, et al., 1992), and cognition (Bressler & Kelso, 2001). Applications of dynamical models to decision-making under uncertainty have focused on either micro-level or macro-level behavioral observations. For example Brown & Holmes (2001) modeled a simple choice task using a dynamical model of firing rates of neurons. On a macro-level, we find examples of dynamical models of multi-agent decision-making processes (for a brief overview, see Lu, Chen & Yu, 2011).

A One-Dimensional Model of Multi-Stability and Hysteresis in Risky Choice

To model the observed switching between *R* and *S*, we propose a nonlinear dynamical system that has previously been applied to other cases in which individuals switched between two different behaviors, and where nonlinear phenomena like hysteresis and reversed hysteresis informed the use of a nonlinear dynamical model (e.g., Tuller et al., 1994). Equation 1 gives the potential function of the one-dimensional model.

$$V(x) = kx - \frac{x^2}{2} - \frac{x^4}{4} + \xi, \qquad (1)$$

where x is the observed choice, k the control parameter, and a noise term ξ is added to each choice.

A potential function is the integral of the differential equation describing the evolution of the state variable x (in our case, the observed choice), which means that a minimum or maximum of the potential function corresponds to a stable state of the system. Our system's potential function therefore reveals the *attractor* and *repeller states*, to which the system is attracted to or repelled from (see Kelso, 1995 and Strogatz, 2000 for more background on dynamical systems). The behavior of our dynamical system is driven by a control parameter k.

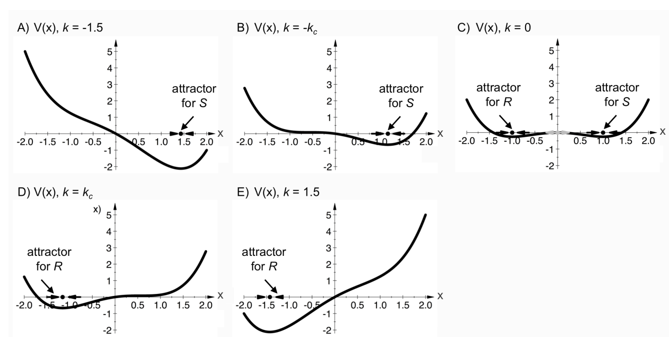

Figure 5: Potential landscape for five different values of k. Depending on the direction of change, a phase transition occurs between the two possible attractors for a critical value of k, $\pm k_c$.

Figure 5 shows some examples of the shape of the potential function, or *attractor landscape*, for different values of k. For a critical value of k, a *bifurcation* occurs (for both $k = k_c$, and $k = -k_c$), causing a *phase transition* between risk-seeking and risk-avoiding choices or vice versa. A phase transition occurs for a different value of k, depending on the direction of change, which explains hysteresis. By defining the two attractor states as the choice for *R* and *S* respectively, this model thus explains switches between risk-seeking and risk-avoiding choices, as well as multi-stability through hysteresis (although not reversed hysteresis; see below for a more detailed discussion of reversed hysteresis).

Parameter Selection and Optimization The potential function offers a way to simulate sequential choice data. The key to modeling the risky choice phenomena is the control parameter *k*, which has to reflect the changing risk in *R*. We propose *k* as a simple linear function of the risk in *R* at choice *j* and a baseline individual value, k_0, such that

$$k_j = k_0 - R_j, \tag{2}$$

By sampling k_0 from a uniform distribution spanning all possible values of *k* between two extremes, and using Eq. (1) and (2), we simulated an entire range of possible choice data. The lower boundary for k_0 corresponds to the case where only the attractor for *S* exists, regardless of the value of the risk in *R*, and the upper boundary corresponds with only one attractor for *S*. Using a bootstrapped optimization with respect to the difference between the simulated and empirical choices on the *DI* and *ID* sequences of our main experiment (no variability in step-size, no speed manipulation), we were able simulate 86% of the observed choices. The differences in switch-points for reversed hysteresis are relatively small compared to the total range of values for *R* (*M* = \$170.45, *SD* = \$183.08). This explains that, although the model does not account for reversed hysteresis, it generates a high proportion of correct choices.

Individual Risk Sensitivity A frequently reported result in research on decision-making under uncertainty is that people have relatively static personality characteristics that determine their risk-taking behavior (e.g. Mishra & Lalumière, 2011). Accordingly, we hypothesize that k_0, the individual baseline value of the control parameter *k* reflects risk sensitivity or propensity, and should therefore closely relate to participants' BART scores. Correlation between the participants' BART scores and the optimal values of k_0 however is very low, $r(33) = -.15$, $p = .36$ for the *ID* sequences, and $r(33) = -.04$, $p = .81$ for the *DI* sequences.

Modeling Reversed Hysteresis The current model does not account for reversed hysteresis, while up to 42% of participants show reversed hysteresis in their choice behavior. Lopresti-Goodman, Turvey, & Frank (2012) provide a way to extend nonlinear dynamical models that includes reversed hysteresis using an *auto-regulated* control parameter. Negative auto-regulation forces the dynamical system to remain close to the bifurcation line and may reflect habituation to the amount of risk presented in the choices; rendering the choice for *S* or *R* unstable. This would also explain why the amount of hysteresis relative to reversed hysteresis increases with more variability in the increments of *R* (Figure 4). Larger variability interferes with the habituation and diminishes the effect of negative auto-regulation.

Discussion

There are many models of risky choice (see Glöckner & Pachur, 2012 for a review). However, in order to account for multi-stability, nonlinearity is a necessary assumption. The results presented here show multi-stability in risky choice, for which we have provided a basic nonlinear dynamical model. The model provides a way to explain decision-making under uncertainty within the framework of complexity theory; a relative newcomer to the social sciences that offers a promising new perspective on human cognition (Van Orden, Holden, & Turvey, 2003). Although the current model does not explain reversed hysteresis, it does provide a blueprint for a nonlinear dynamical model that can capture the entire range of observed choice behavior.

The aim of modeling was to provide a formal description of the observed decision-making behavior. Moreover, our hope is that identifying the right kind of nonlinear models will eventually lead to insights into the underlying processes or mechanisms. One of the strengths of the model is that multi-stability is an inherent behavior of the nonlinear dynamical system, pre-empting the need for weight functions or exceptions. The model also provides a starting point for theorizing about the psychological processes underlying the behavior. The control parameter is a single parameter that captures the switching between risk-seeking and risk-averse choices. Unexpectedly, however, there was no correlation between participants' BART scores and the baseline value of the control parameter, k_0. Upon reflection, this result is not as surprising after all. Nonlinear dynamical systems are especially useful in capturing change and the phenomena that are associated with change, like hysteresis. The BART however assumes individual risk preference is a temporarily static personality trait. The current results therefore indicate that risk preference is a highly complex and multi-dimensional construct and that the dynamics of subsequent risky choice behavior cannot be captured in a single measure of risk sensitivity.

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