

A Quantum Probability-theoretic account of human judgment using Positive-Operator-Valued Measures

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Abstract

People make logically inconsistent probability judgments. The “Linda” problem is a well-known example, which often elicits a conjunction/disjunction fallacy: probability of constituent event A (B) judged more/less likely than their conjunction/disjunction. The Quantum Judgment model (QJM, Busemeyer et al 2011) explains such errors, which are not explainable within classical probability theory. We propose an alternative axiomatic approach in the framework of quantum probability theory that employs *positive* operators representing the set of general queries, in contrast to QJM which uses *projection* operators. Like QJM, our model accounts for conjunction/disjunction fallacies, averaging type errors, and unpacking effects, suggesting that it provides a viable model of judgement error. Further differences between our model and QJM are also discussed.

Keywords: Probability judgment; Quantum Probability theory; conjunction/disjunction fallacy; “Linda” problem.

Introduction

People make probability judgments that are logically inconsistent with classical probability theory. The “Linda” problem is a well-known example: Participants are told that Linda was a philosophy student and an anti-nuclear supporter, and asked to judge her most likely current situation as either (a) feminist supporter, (b) bank teller, (c) feminist and bank teller—conjunction, (d) feminist, but not bank teller, or (e) feminist or bank teller—disjunction. Judging (b) as more likely than (c) is a *conjunction error* (fallacy), since by classical probability $Prob(A \text{ and } B) \leq Prob(B)$; judging (a) as more likely than (e) is a *disjunction error*, since $Prob(A) \leq Prob(A \text{ or } B)$. These fallacies are well-known (Tversky & Kahneman, 1983; Bar-Hillel & Neter, 1993), but they are not explained by classical models (Busemeyer, Pothos, Franco, & Trueblood, 2011).

Busemeyer et al. (2011) proposed an alternative model based on *quantum probability theory* (QPT) (Peres, 1993). Their quantum judgment model (QJM) uses the properties of *quantum coherence* and *quantum interference* to explain conjunction and disjunction errors, respectively. An explanation of QJM follows (Appendix A summarizes QPT).

Using the Linda problem as an example, QJM assumes that beliefs about states of the world (e.g., *Linda is a feminist*) are represented as vectors Ψ in a *Hilbert space* \mathcal{H} where, e.g., the basis vectors represent feature combinations (e.g., non-/feminist, young/old, gay/straight). An event E (e.g., corresponding to the proposition *Linda is a bank-teller*) is

a *projection operator*, which projects the belief vector onto a subspace representing a possible outcome: e.g., *Yes* is a possible outcome (subspace) for *Linda is a bank-teller*. A *projection operator* E applied to Ψ , written $E\Psi$ (i.e. vector/matrix multiplication), returns the probability of belief in that outcome, computed as $Prob(E) = \langle \Psi | E \Psi \rangle$, where $\langle \cdot | \cdot \rangle$ is *Dirac notation* for the inner product. For events corresponding to conjunctions of propositions “ E and F ” (e.g., Linda is a feminist and Linda is a bank-teller), the belief in an outcome is computed as: if $Prob(E) > Prob(F)$, then $Prob(E \text{ and } F) = \langle \Psi | E F E \Psi \rangle$, else if $Prob(E) < Prob(F)$, then $\langle \Psi | F E F \Psi \rangle$.¹

In this paper, we provide an alternative formulation of human probability judgment within the framework of quantum probability theory. The most general class of queries is represented by the space of *positive operators*, which includes projection operators. Motivated by this observation, we propose a set of axioms to define a positive operator corresponding to the conjunction of a pair of general propositions. We provide an example that is consistent with this set of axioms. Moreover, we also show how this reformulation accounts for the conjunction and disjunction fallacies, averaging type errors, and unpacking effects.

Quantum formulation using positive operators

A quantum system is described by a Hilbert space (see Appendix A). In quantum theory, a general query (or event) is represented by an operator A satisfying $\mathbf{0} \leq A \leq \mathbf{1}$. It should be noted that a projection operator also satisfies this condition. Thus an observable which takes a value in a set Ω is represented by a positive-operator-valued measure (POVM) $\{A_a\}_{a \in \Omega}$ on Ω . Roughly, a POVM can be regarded as a “fuzzy” version of projection-valued measure (PVM). Thus a POVM is often called an unsharp observable.

Assumption 1 *A person’s belief state is described by a state of a quantum system.*

We denote a Hilbert space by \mathcal{H} .

Assumption 2 *An event that has a family of possible outcomes Ω is described by a family of positive operators $E = \{E_i\}_{i \in \Omega}$ on \mathcal{H} that satisfies $\sum_{i \in \Omega} E_i = \mathbf{1}$. (Such a family of positive operators is called a positive-operator-valued measure (POVM).)*

¹ $Prob(E \text{ and } F)$ is undefined when $Prob(E) = Prob(F)$.

In the above formulation, an operator corresponding to a proposition “ A and B ” is not specified. To give quantitative predictions, however, this operator needs to be specified. We take an axiomatic approach to identify a suitable operator. Let us denote the operator corresponding to “ A and B ” by $\Lambda(A, B)$. We assume that for a pair of POVMs $\{A_a\}$ and $\{B_b\}$, an operator corresponding to “ A_a and B_b ” does not depend on A_c ’s ($c \neq a$) and B_d ’s ($d \neq b$). That is, Λ is defined as a map $\Lambda : E_+(\mathcal{H}) \times E_+(\mathcal{H}) \rightarrow E_+(\mathcal{H})$, where $E_+(\mathcal{H}) := \{A \mid \mathbf{0} \leq A \leq \mathbf{1}\}$. It is natural to suppose that this Λ satisfies the following conditions:

- (o.b) $\Lambda(A, B)$ satisfies $\mathbf{0} \leq \Lambda(A, B) \leq \mathbf{1}$ for any $A, B \in E_+(\mathcal{H})$.
- (i.b) For any POVMs $\{A_a\}$ and $\{B_b\}$, it holds that $\sum_{a,b} \Lambda(A_a, B_b) = \mathbf{1}$. (Thus, $\{\Lambda(A_a, B_b)\}$ becomes a POVM.)
- (ii.b) $\Lambda(A, A) = A$ for any projection A .
- (iii.b) $\Lambda(A, \mathbf{1}) = A$ for any $A \in E_+(\mathcal{H})$.
- (iv.b) $\Lambda(UAU^*, UBU^*) = U\Lambda(A, B)U^*$ for any $A, B \in E_+(\mathcal{H})$ and any unitary operator U .

Some comments are helpful to understand each condition. Condition (o.b) is necessary to guarantee that the framework is closed under conjunction and well-defined. Condition (i.b) means that summation of the probabilities “ A_a and B_b ” for running a and b is 1. Condition (ii.b) represents a trivial requirement. A proposition “Linda is a feminist and Linda is a feminist” is equivalent to “Linda is a feminist”. A restriction of A in condition (ii.b) may seem strange. However, even in a classical system, confirming that a fuzzy query is true does not guarantee that the same query is true. Therefore we impose a weaker condition than the one above. $\mathbf{1}$ in condition (iii.b) represents a trivial proposition such as “Linda is Linda”. This condition implies that the proposition “Linda is a feminist and Linda is Linda” is equivalent to “Linda is a feminist”. Condition (iv.b) may need a detailed explanation. It means that an operator corresponding to “ A and B ” should be determined only by the inter-relationship between A and B . In quantum theory, the relationship between A and B is exactly the same as that between UAU^* and UBU^* because unitary operation U can be interpreted as something like a “coordinate transformation”. Thus $\Lambda(UAU^*, UBU^*)$ should be written as a function of $\Lambda(A, B)$ and U . This function $f(\Lambda(A, B), U) := \Lambda(UAU^*, UBU^*)$ must satisfy $f(\Lambda(A, B), UV) = f(f(\Lambda(A, B), V), U)$. Now we have $f(\Lambda(U^*AU, \mathbf{1}), U) = \Lambda(UU^*AU^*U, \mathbf{1}) = \Lambda(A, \mathbf{1})$. Using condition (iii.b), we obtain for any A and U , $f(U^*AU, U) = A$. Setting $U^*AU = B$ for an arbitrary B , we obtain

$$f(B, U) = UBU^*.$$

Thus it holds that $\Lambda(UAU^*, UBU^*) = f(\Lambda(A, B), U) = U\Lambda(A, B)U^*$. Note that these conditions are rather weak. For

instance, we do not require “ A and B ” to be equivalent with “ B and A ”.

Before showing the existence of a Λ satisfying these requirements, we show a proposition easily derived from them.

Proposition 1 Suppose Λ satisfies the requirements (o.b) - (iv.b). It holds that for any projections P and Q satisfying $P + Q \leq \mathbf{1}$, $\Lambda(P, Q) = \mathbf{0}$, and for any A with $\mathbf{0} \leq A \leq \mathbf{1}$, $\Lambda(A, \mathbf{0}) = \mathbf{0}$.

Proof: Let us begin with $\Lambda(P, Q) = 0$ for projections P and Q with $P + Q \leq \mathbf{1}$. We can define a PVM $\{A_0, A_1, A_2\} := \{P, Q, \mathbf{1} - P - Q\}$, Considering $\sum_{a,b} \Lambda(A_a, A_b) = \mathbf{1}$, we obtain

$$\begin{aligned} & P + Q + (\mathbf{1} - P - Q) + \Lambda(P, Q) + \Lambda(Q, P) \\ & + \Lambda(P, \mathbf{1} - P - Q) + \Lambda(\mathbf{1} - P - Q, P) \\ & + \Lambda(Q, \mathbf{1} - P - Q) + \Lambda(\mathbf{1} - P - Q, Q) = \mathbf{1}, \end{aligned}$$

where we used Conditions (i.b) and (ii.b). It concludes $\Lambda(P, Q) = \mathbf{0}$.

Consider a POVM $\{A, \mathbf{1} - A\}$ and $\{\mathbf{1}, \mathbf{0}\}$. Condition (i.b) and (iii.b) are used to show

$$\begin{aligned} & \Lambda(A, \mathbf{1}) + \Lambda(A', \mathbf{1}) + \Lambda(A, \mathbf{0}) + \Lambda(A', \mathbf{0}) \\ & = A + A' + \Lambda(A, \mathbf{0}) + \Lambda(A', \mathbf{0}) = \mathbf{1}. \end{aligned}$$

It concludes $\Lambda(A, \mathbf{0}) = \mathbf{0}$. ■

To illustrate the existence of Λ , let us consider the following example.

Example 1 Fix $0 \leq p \leq 1$. For any A, B satisfying $\mathbf{0} \leq A \leq \mathbf{1}$ and $\mathbf{0} \leq B \leq \mathbf{1}$, we define $\Lambda_p(A, B)$ by

$$\Lambda_p(A, B) = pA^{1/2}BA^{1/2} + (1-p)B^{1/2}AB^{1/2}.$$

Using $\mathbf{0} \leq B \leq \mathbf{1}$, one can show $\mathbf{0} \leq A^{1/2}BA^{1/2} \leq \mathbf{1}$. Thus $\mathbf{0} \leq \Lambda_p(A, B) \leq \mathbf{1}$ holds and condition (o.b) is satisfied. Let us examine condition (i.b). Consider a pair of POVM $\{A_a\}$ and $\{B_b\}$. We obtain

$$\begin{aligned} & \sum_{a,b} \Lambda_p(A_a, B_b) \\ & = \sum_{a,b} \left(pA_a^{1/2}B_bA_a^{1/2} + (1-p)B_b^{1/2}A_aB_b^{1/2} \right) \\ & = p \sum_a A_a^{1/2} \sum_b B_b A_a^{1/2} + (1-p) \sum_b B_b^{1/2} \sum_a A_a B_b^{1/2} \\ & = p \sum_a A_a + (1-p) \sum_b B_b = \mathbf{1}. \end{aligned}$$

Condition (ii.b) is satisfied because $P^{1/2}PP^{1/2} = P$ holds for a projection P . Condition (iii.b) also follows immediately. In addition, it holds that

$$\begin{aligned} & \Lambda_p(UAU^*, UBU^*) \\ & = pUA^{1/2}U^*UBU^*UA^{1/2}U^* \\ & + (1-p)UB^{1/2}U^*UAU^*UB^{1/2}U^* \\ & = U\Lambda_p(A, B)U^*, \end{aligned}$$

where we used $U^*U = \mathbf{1}$. Thus condition (iv.b) is satisfied. .

Thus we proved the following theorem.

Theorem 1 *There exists Λ satisfying Conditions (o.b) - (iv.b). (This Λ is not uniquely determined.)*

Conjunction and Disjunction Fallacies

The remaining task is to show that there exists a Λ that accounts for the conjunction and disjunction fallacies. We take $\Lambda_{1/2}$ introduced in Example 1.

Let us consider a model described by a two-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$ which has an orthonormalized basis e_0 and e_1 . A pair of PVMs $A = \{A_0, A_1\}$ and $B = \{B_0, B_1\}$ are defined as $A_n = |e_n\rangle\langle e_n|$ and $B_n = |f_n\rangle\langle f_n|$ for $n = 0, 1$, where f_0 and f_1 are defined by

$$\begin{aligned} f_0 &:= \frac{1}{\sqrt{2}}(e_0 + e_1) \\ f_1 &:= \frac{1}{\sqrt{2}}(e_0 - e_1). \end{aligned}$$

Let us consider a pure state described by a vector

$$\Psi := \sqrt{\frac{9}{10}}e_1 - \sqrt{\frac{1}{10}}e_0.$$

The probability for each proposition is calculated as, $Prob(A_1) = \frac{9}{10}$, $Prob(A_1 \text{ or } B_0) = 1 - Prob(\Lambda_{1/2}(A_0, B_1)) = \frac{31}{40}$, $Prob(A_1 \text{ and } B_0) = Prob(\Lambda_{1/2}(A_1, B_0)) = \frac{11}{40}$, and $Prob(B_0) = \frac{1}{5}$. They satisfy

$$Prob(A_1) > Prob(A_1 \text{ or } B_0) > Prob(A_1 \text{ and } B_0) > Prob(B_0).$$

Thus this example shows both conjunction and disjunction fallacies. Note that conjunction and disjunction fallacies are supported by other choices of Ψ . It is an important future work to identify the relevant states.

In addition, Λ_p ($0 < p < 1$) given by Example 1 is consistent with an observation of averaging type errors (Fantino, Kulik, & Stolarz-Fantino, 1997). Consider two general propositions A and B . Suppose that a state Ψ satisfies $Prob(A) = \langle \Psi | A \Psi \rangle > \langle \Psi | B \Psi \rangle = Prob(B)$. Then $Prob(A) > Prob(A \text{ and } B)$ must follow. In fact, in our model it holds that

$$\begin{aligned} &Prob(A \text{ and } B) \\ &= \langle \Psi | \Lambda_p(A, B) \Psi \rangle \\ &= p \langle \Psi | A^{1/2} B A^{1/2} \Psi \rangle + (1-p) \langle \Psi | B^{1/2} A B^{1/2} \Psi \rangle \\ &\leq p \langle \Psi | A \Psi \rangle + (1-p) \langle \Psi | B \Psi \rangle \\ &< \langle \Psi | A \Psi \rangle = Prob(A). \end{aligned}$$

where we used $B \leq \mathbf{1}$ and $A \leq \mathbf{1}$.

Unpacking effect, in its broad sense, is interpreted as a difference between $Prob(A \text{ and } B) + Prob(A \text{ and } B')$ and $Prob(A)$ (Rottenstreich & Tversky, 1997). That is, the law of classical (Kolmogorov) probability,

$$Prob(A \text{ and } B) + Prob(A \text{ and } B') = Prob(A)$$

is violated. We can show that this effect inevitably occurs between noncommutative sharp propositions no matter how we set Λ .

Theorem 2 *Let P and Q be propositions represented by projection operators. If there is no state violating*

$$\begin{aligned} Prob(P \text{ and } Q) + Prob(P \text{ and } Q') &= Prob(P) \\ Prob(P \text{ and } Q) + Prob(P' \text{ and } Q) &= Prob(Q), \end{aligned}$$

P and Q commute with each other.

Proof: If the above equations hold for arbitrary states, Λ satisfies

$$\begin{aligned} \Lambda(P, Q) + \Lambda(P, Q') &= \mathbf{1} \\ \Lambda(P, Q) + \Lambda(P', Q) &= \mathbf{1} \end{aligned}$$

and vice versa. These equations mean that PVMs $\{P, \mathbf{1} - P\}$ and $\{Q, \mathbf{1} - Q\}$ are jointly measurable. Hence, P and Q commute with each other (Miyadera, 2011). \blacksquare

Moreover, for general propositions, we have the following theorem.

Theorem 3 *Let A and B be general propositions. If there is no state violating*

$$\begin{aligned} Prob(A \text{ and } B) + Prob(A \text{ and } B') &= Prob(A) \\ Prob(A \text{ and } B) + Prob(A' \text{ and } B) &= Prob(B), \end{aligned}$$

their intrinsic ambiguities defined by $V(A) = \|A - A^2\|$ satisfy

$$V(A)^{1/2}V(B)^{1/2} \geq \frac{1}{2}\|[A, B]\|,$$

where an operator norm $\|\cdot\|$ is defined by $\|A\| := \sup_{\Psi \neq 0} \frac{\|A\Psi\|}{\|\Psi\|}$.

This theorem was proved in Miyadera and Imai (2008).

Discussion

In this paper, we provided an axiomatic formulation of human probability judgment within the general framework of quantum probability theory. A concrete instantiation was found that satisfies the axioms while accounting for the conjunctive/disjunctive fallacies, averaging type errors, and unpacking effects. We note, though, that QJM accounts for other effects that we have not yet addressed, e.g., *order effect*.

Here, we comment on some differences between our approach and QJM. In contrast to Busemeyer's model, our POVM formalism does not require computing $Pr(A)$ (nor $Pr(B)$) to obtain $Pr(A \text{ and } B)$ because $\Lambda(A, B)$ does not depend on a state. Also, in Busemeyer's formalism an exhaustive set of conjunctions may not sum to 1 (see Appendix B), whereas in our formalism the summation of probabilities is set to 1 (see Axiom (i.b)). Finally, our formulation can be naturally generalized to use mixed states. Further work is needed to explore the implications of these differences.

Proponents of QPT-based approaches divorce themselves from a commitment to the brain as a quantum device (Busemeyer et al., 2011). As a *descriptive* theory of human

judgment, one need not be committed to a quantum mechanical implementation. However, if a *causal* theory is sought—ultimately so for a science of cognition, then a theory based on QPT must be reconciled against the (lack of) evidence showing that the brain is indeed a quantum device (but, see Hameroff, 2002). An alternative to this predicament is to seek yet a further generalization of the quantum framework, which does not depend on quantum mechanics. *General operational probability theory* (Dvurecenskij & Pulmannova, 2000) and *category theory* (MacLane, 2000) are two possibilities for future investigation.

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Appendix

Appendix A: Quantum Probability theory

A *quantum system* is described by a *Hilbert space* \mathcal{H} , which is a vector space (which we assume to be finite dimensional, i.e., $\dim \mathcal{H} < \infty$) over the complex field \mathbf{C} that is equipped with an *inner product*. The inner product $\langle \cdot | \cdot \rangle$ defines a map $\mathcal{H} \times \mathcal{H} \rightarrow \mathbf{C}$ satisfying: (i) $\langle \phi | c_1 \psi_1 + c_2 \psi_2 \rangle = c_1 \langle \phi | \psi_1 \rangle + c_2 \langle \phi | \psi_2 \rangle$ for all $c_1, c_2 \in \mathbf{C}$ and $\phi, \psi_1, \psi_2 \in \mathcal{H}$; (ii) $\langle \phi | \psi \rangle = \langle \psi | \phi \rangle^*$ for all $\phi, \psi \in \mathcal{H}$; and (iii) $\langle \phi | \phi \rangle \geq 0$, and $\langle \phi | \phi \rangle = 0 \Leftrightarrow \phi = 0$. A family of vectors $\{e_i\}_{i=1}^{\dim \mathcal{H}}$ is called an orthonormalized basis of \mathcal{H} , if it satisfies $\langle e_i | e_j \rangle = 0$ for $i \neq j$ and $\langle e_i | e_i \rangle = 1$ for all i . A linear map $A : \mathcal{H} \rightarrow \mathcal{H}$ is called an *operator*. Every operator A is associated with a unique operator A^* satisfying $\langle A^* \psi | \phi \rangle = \langle \psi | A \phi \rangle$ (Riesz theorem); A^* is called a *conjugate operator* of A . An operator U satisfying $UU^* = U^*U = \mathbf{1}$ is called a *unitary operator*. An operator A satisfying $A = A^*$ is called a *self-adjoint operator*. A self-adjoint operator P satisfying $P = P^* = P^2$ is called a *projection operator*. A self-adjoint operator A satisfying $\langle \psi | A \psi \rangle \geq 0$ for all $\psi \in \mathcal{H}$ is called a *positive operator*, written as $A \geq \mathbf{0}$, where $\mathbf{0}$ denotes a null operator. Every projection operator is a positive operator. For a positive operator A , $A^{1/2}$ is defined as a unique positive operator satisfying $A^{1/2} A^{1/2} = A$.

State and observable are central notions in any physical theory. In quantum theory, a state is represented by a self-adjoint operator ρ , called a *density operator*, satisfying: (i) $\rho \geq \mathbf{0}$; and (ii) $\text{tr}(\rho) = \sum_i \langle e_i | \rho e_i \rangle = 1$ for any orthonormal basis— tr is called *trace*. The set of all states is *convex*: i.e., any combination of two states $p\rho + (1-p)\sigma$ (for $0 \leq p \leq 1$ and states ρ, σ) is also a state. A state ρ that does not have a nontrivial decomposition is called a *pure state*. A pure state is represented by a projection operator whose *rank* is 1, i.e., there exists a unit vector ψ ($\|\psi\| = 1$) satisfying $P\phi = \psi \langle \psi | \phi \rangle$. (P is also written $|\psi\rangle\langle\psi|$.) This correspondence allows one to identify a unit vector with a pure state. A state that is not pure is called *mixed*.

An observable which takes a value in a set Ω (assumed to be discrete set in this paper) is described by a family of positive operators $\{A_a\}_{a \in \Omega}$ satisfying $\sum_{a \in \Omega} A_a = \mathbf{1}$. This is called a *positive-operator-valued measure* (POVM). The

probability of an outcome $a \in \Omega$ in a state ρ is given by $\text{Prob}(A_a) = \text{tr}(\rho A_a)$. A POVM $\{A_a\}$ is called a *projection-valued measure* (PVM) if A_a is a projection operator for each a . PVMs are often treated as more fundamental objects because each POVM can be represented as a PVM in an enlarged space (Naimark extension theorem). In fact, the space of projection operators can be regarded as a generalization of the Boolean algebra.

Appendix B: Summation of the probabilities may not agree with one in Busemeyer et al. (2011).

Suppose that E and F (and their negations $E' = \mathbf{1} - E$ and $F' = \mathbf{1} - F$) satisfy for a state ψ , $\langle \psi | F \psi \rangle > \langle \psi | E \psi \rangle > \langle \psi | E' \psi \rangle > \langle \psi | F' \psi \rangle$. Then we obtain $\text{Prob}(E \text{ and } F) = \langle \psi | FEF\psi \rangle$, $\text{Prob}(E' \text{ and } F) = \langle \psi | F'E'F\psi \rangle$, $\text{Prob}(E' \text{ and } F') = \langle \psi | E'F'E'\psi \rangle$. Their summation may not agree with 1. In fact, let us consider $\mathcal{H} = \mathbf{C}^2$ with an orthonormalized bases $\{e_0, e_1\}$ and projection operators E and F defined by $E = |e_1\rangle\langle e_1|$ and $F = |f_0\rangle\langle f_0|$, where f_0 is defined by $f_0 := \sqrt{\frac{1}{2}}e_1 + \sqrt{\frac{1}{2}}e_0$. It can be shown that a state $\psi = \sqrt{\frac{1}{4}}e_1 - \sqrt{\frac{3}{4}}e_0$ satisfies the above inequality, giving

$$\begin{aligned} \langle \psi | FEF\psi \rangle + \langle \psi | F'E'F\psi \rangle + \langle \psi | EF'E\psi \rangle + \langle \psi | E'F'E'\psi \rangle \\ = 1 + \frac{\sqrt{3}}{4}. \end{aligned}$$

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