

Mathematical Cognition as Embodied Simulation

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Abstract

Based on behavioral, neuroimaging and neuropsychological data, I argue that a key to understanding mathematical cognition is the sharing of neural resources between sensorimotor and mathematical processes. Mathematical cognition is embodied in the sense that it is grounded in simulations of sensorimotor processes through the use of neural resources that are also active in bodily perception and action. There are two approaches to the study of embodied mathematical cognition: Behavioral, neuroimaging and neuropsychological investigations providing empirical evidence, and the study of conceptual metaphors, focusing on how inferences from physical domains are used to understand abstract mathematical ideas. The first approach suffers from not providing a unified explanation, while the second approach is criticized for not having empirical validation. I discuss the possible implications of approaching to mathematical cognition as embodied simulation in relating disparate findings to provide a more connected picture of how mathematics emerges from the embodied mind.

Keywords: embodiment; embodied cognition; mathematical cognition; simulation theories

Embodied cognition is a theoretical stance that argues that cognitive processes are grounded in the body's interaction with the world. Different approaches in embodied cognition propose varying levels for bodily involvement in higher cognition. Clark (1999) has distinguished between simple versus radical embodiment. Simple embodiment focuses on how the body and environment places constraints on a theory of inner organization and processing. Radical embodiment, however, asserts that all cognitive processes are grounded in the sensorimotor system, proposing a profound change in the "subject matter and theoretical framework of cognitive science" (p. 348). The fundamental difference between these two approaches is that simple embodiment still relies on internal representations, especially in explaining higher level thinking, whereas radical embodiment entirely rejects the idea of an internal realm and provides a *representation free* account of cognitive phenomena. I use the term *simulation theories of cognition* to refer to theories positing that all cognitive processes are simulations of sensorimotor processes. Note that the term *simulation theories* is also used to refer to a theory of mind asserting that humans understand other people's mental states by adopting their perspective (Davies & Stone, 1995), which is different than the usage here.

Simulation theories posit a decoupling of sensorimotor functions from their original physical inputs and outputs. For example, consider the case of counting on one's fingers. In its initial form counting can be done through explicit motor behavior where an observer can see the fingers moving. However, the motor movement of fingers can become gradually more subtle, where at some point it might merely seem like twitching to the observer. We can push the activity inward even further allowing the use of motor programs without any overt behavior. At this point finger counting is a motor simulation. This situation exemplifies how a motor function, without overt behavior, can be the underlying neural mechanism for off-line thinking in the very simple case of counting (Wilson, 2002).

Previous theories focused on how conceptual content is represented in the sensorimotor system. Gallese & Lakoff (2005) proposed that embodied simulations are the source of both structural and semantic content in conceptual knowledge. Embodied simulations take place in multimodal sensorimotor networks. Unlike the conventional idea of distinct sensory and motor areas communicating through association areas, multimodality refers to the integration of sensory modalities with one another and also with motor modalities. Barsalou (1999) argued that during perceptual experience association areas in the brain capture bottom-up sensory-motor patterns. Later, during the use of perceptual symbols association areas facilitate some of the same sensory-motor areas in a top down manner. Through experience, memories of the same component are stored in a schematic manner. The memories implement simulators of the perceptual experiences they represent. Simulators can be perceptual, proprioceptive, or introspective. Abstract concepts are grounded in the combinatorial and recursive integration of simulators.

Mathematics is often characterized as a challenge to embodiment (Nunez, 2008). Although it is relatively difficult to apply the idea of embodied simulations to explain mathematical cognition due to abstract nature of mathematics, there is accumulating evidence for how basic mathematical processes are grounded in the sensorimotor system. In this paper I review different studies on mathematical cognition and discuss some of the challenges in interpreting findings to create a meaningful image of how mathematics can emerge from the embodied mind.

Embodiment of Mathematical Thinking

Research on embodiment of mathematics is still in its infancy. Mathematical cognition is a big puzzle with many pieces, each piece requiring us to draw knowledge from a

different field. Currently, there are two trends in studying embodiment of mathematical cognition: 1) empirical investigations of basic number processing skills, for example number recognition and comparison, parity judgment, arithmetic and to some extent simple algebra through behavioral, neuroimaging and neuropsychological studies. Another trend, most typically exemplified by Lakoff and Johnson's book on embodied mathematics (Lakoff & Nunez, 2000), is the study of conceptual metaphors in mathematics to explain how mathematical concepts are grounded in bodily processes. Both trends have strengths and shortcomings. Empirical studies provide accumulating disparate evidence on embodiment of number processing; however they do not provide a unified, big picture of how number processing is grounded in the sensorimotor system. Nevertheless, general theories explaining how number processing takes place in the brain exists. One, arguably the most well-known, theory is the triple-code model (Dehaene, Piazza, Pinel, & Cohen, 2003), which provides a relatively disembodied account of number processing.

The second trend is explanation of mathematical cognition based on conceptual metaphors (Lakoff & Nunez, 2000). The role of conceptual metaphors in language and thinking was first studied by Lakoff and Johnson (1980). Sfard (1994) incorporated ideas from cognitive linguistics, on the use of metaphors in language and thinking (Lakoff & Johnson, 1980) to explain how we rely on daily physical inferences to make sense of mathematical concepts. Lakoff and Nunez (2000) extended this program by inquiring how metaphors are used in diverse domains of mathematics, including for example algebra, logic, sets and even trigonometry. The main argument in this approach is that we use inferences from our bodily interactions to understand mathematical concepts; a conceptual metaphor links a physical source domain to a target abstract domain. This approach is criticized for lacking empirical verification and for overextending the claims of embodiment to higher domains of mathematics without sufficient support (Goldin, 2001).

I believe that there is a need for bridging these two trends to have a unified explanation of numerical cognition that is supported by empirical findings. Approaching to mathematical cognition as embodied simulations might have the potential to do that.

Empirical Evidence

There are four major sources of evidence supporting the relation between bodily processes and mathematical cognition. First, studies on neural correlates of hand movements and action understanding of hand gestures point to an overlapping circuitry in the prefrontal and intraparietal regions with number processing (Binkofski et al., 1999; Chong, Cunnington, Williams, Kanwisher, & Mattingley, 2008; Corina & Knapp, 2006; Peltier et al., 2007; Sakata & Taira, 1998). In addition, a separate body of neuroimaging research point to a relation between neural correlates of hand/finger movement control and number processing

(Andres, Seron, & Olivier, 2007). Secondly, studies conducted with repetitive Transcranial Magnetic Stimulation (rTMS) show excitability of hand muscles during different number processing tasks (Andres, et al., 2007; Sato, Cattaneo, Rizzolatti, & Gallese, 2007). Third, behavioral studies on math learning provide evidence for a) better math learning when instruction is supported with hand gestures, b) higher problem solving performance when non-communicative hand gestures are allowed, compared to when hands are restricted, and c) non-communicative hand gestures during problem solving provide clues for misconceptions in conceptual understanding of arithmetic and algebra (Goldin-Meadow, 1997, 1999, 2006; Goldin-Meadow, Nusbaum, Kelly, & Wagner, 2001; Goldin-Meadow & Singer, 2003; Goldin-Meadow & Wagner, 2005). The fourth major support comes from neuropsychological conditions, particularly Gerstmann syndrome (Gerstmann, 1940), which is discussed later in this paper.

Conceptual Metaphors and First Person Accounts

The use of conceptual metaphors is characterized by the use of a physical, source domain to understand an abstract, target domain. First person accounts of mathematical experience provide additional insight about how metaphorical thinking is involved in mathematical processes. In a letter to mathematician Jacques Hadamard, Einstein once wrote:

Thoughts do not come in any verbal formulation.

Words and language, whether written or spoken, do not seem to play any part in my thought processes. The psychological entities that serve as building blocks for my thought are certain signs or images, more or less clear, that I can reproduce and recombine at will... The above mentioned elements are, in my case, of visual and some of muscular type. Conventional words or other signs have to be sought for laboriously only in a secondary stage, when the mentioned associative play is sufficiently established and can be reproduced at will (Hadamard, 1945, pp. 142,143)

Sfard questioned mathematicians about how they process mathematical concepts. In particular, she investigated if they process mathematical concepts in a way that is similar to physical objects. When Sfard asked how it feels to have a deep understanding of a mathematical idea, three mathematicians responded by saying, "identify a structure [one is] able to grasp somehow", "to see an image", and "to play with some unclear images of things". One mathematician reported, "In those regions where I feel an expert ... the concepts, the [mathematical] objects turned tangible for me" (Sfard, 1994, p. 48). Another mathematician stated:

To understand a new concept I must create an appropriate metaphor. A personification. Or a spatial metaphor. A metaphor of structure. Only then can I answer questions, solve problems. I may even be able

then to perform some manipulations on the concept.

Only when I have the metaphor. Without the metaphor I just can't do it. (Sfard, 1994, p. 48)

The same mathematician also reported that the structure he uses has to have some spatial elements no matter how abstract the mathematical idea is. In the same study, mathematicians pointed to personification as another strategy for understanding mathematical concepts. Similarly, Hadamard (1945) reported mathematicians' tendency to treat mathematical concepts as human faces.

In a discussion of understanding and meaning in mathematical thinking, Sfard (1994) distinguished between objectivist and embodied theories of meaning. She characterized objectivist claims about knowledge as propositional and disembodied. Following the steps of Lakoff and Johnson (1980) Sfard defines a metaphor as a relation between a bodily and a conceptual domain. Metaphors facilitate our use of inferences from physical and bodily experiences to understand abstract concepts and relations. Lakoff and Johnson (1980) introduce *embodied schemata* to explain how metaphors work. Embodied schemata is "the vehicle which carries our experimentally constructed knowledge" (Sfard, 1994, p. 46). They are the "... structures of an activity by which we organize our experience in ways that we can comprehend. They are a primary means by which we construct or constitute order and are not mere passive receptacles into which experience is poured" (Lakoff & Johnson, 1980, pp. 29-30). According to Sfard, embodied schemata are non-propositional. They are "... image-like and embodied, embodied in the sense that they should be viewed as analog reflections of bodily experience rather than as factual statements we may wish to check for validity. The non-propositional nature of embodied schemata makes it difficult, sometimes impossible, to describe them in words." In this sense, embodied schemata are preverbal constructs that are dynamic, ever changing and shaped by our physical and social experiences. However, the nature of embodied schemata, how they are shaped in the sensorimotor system and how abstract thinking emerges from these preverbal constructs is still not clear. Although an embodied schema is a preverbal construct shaped in the sensorimotor system, we still talk about it like a cognitive construct since we cannot explain how it relates to the simple bodily functions and sensorimotor interactions.

Interpretation of mathematical thinking as embodied simulations requires a conceptual shift. Mathematical thinking is reconceptualized as simulated sensorimotor activity. This activity takes place in a temporal and spatial stage involving all modalities. As mathematician Alain Connes puts it: "The evolution of our perception of mathematical reality causes a new sense to develop, which gives us access to a reality that is neither visual nor auditory, but something else together" (Dehaene, 1997, p. 149). The key to understanding the multimodal sensorimotor foundations of mathematical might be through

adopting an embodied perspective in designing studies and interpreting data.

An Embodied Approach to Interpreting Neuroimaging Data

Imaging studies, as well as neuropsychological cases, point to the importance of a network of areas consisting of prefrontal and parietal regions, particularly angular gyrus and IPS (Intraparietal Sulcus). In this section I will revisit previous interpretations on the functional contribution of angular gyrus and IPS to number processing and propose an alternative embodied approach to provide a more connected explanation that is also compatible with behavioral and neuropsychological findings. The idea here is to provide an example for how embodied simulations framework can be applied to the interpretation of neuroimaging data in the mathematical cognition domain.

Angular Gyrus

Angular gyrus is located in the inferior parietal cortex. It is situated at a very central location in the cortex, neighboring multimodal sensory regions. It was once characterized as the "association area of association areas" together with the supramarginal gyrus (Geschwind, 1965). Angular gyrus activations, particularly left, were found in various number processing tasks, for example exact addition (Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999), multiplication (Lee, 2000) and number recognition (Pesenti, Thioux, Seron, & Volder, 2000). Although it is established that angular gyrus is an essential part of the number processing network, its role is still not well understood. According to the well-known, triple-code model, being part of the perisylvian language network, the angular gyrus is functional in verbal processing of numbers (Dehaene, et al., 2003). Nevertheless accumulating behavioral, neuroimaging and neuropsychological data tell us a different story about the involvement of angular gyrus.

A relation between angular gyrus and number processing was first formulated when, in 1924, Josef Gerstmann diagnosed a condition, now named Gerstmann's syndrome, with four co-occurring symptoms: finger agnosia (loss in finger sense), acalculia (inability to do simple calculations), left-right disorientation and agraphia (inability to write). Gerstmann found that the condition was most commonly due to a lesion in the left angular gyrus (Gerstmann, 1940). He believed that the main symptom was finger agnosia, a specific type of body schema impairment (autopagnosia) affecting the mental representation of hands and fingers. He proposed that the loss of finger sense combined with the left-right disorientation caused acalculia, - the inability to carry out simple mathematical calculations (Butterworth, 1999b, p. 219). There have been a number of studies reporting data to support Gerstmann's theory. For example, a study examining patients with tumors in and around the angular gyrus found that these patients had impairments in writing, calculating, and finger recognition (Roux, Boetto, Sacko, Chollet, & Tremoulet, 2003). Also, in a rTMS study

of healthy subjects it was found that disruption of the left angular gyrus impaired access to the finger schema and number processing (Rusconi, Walsh, & Butterworth, 2005).

These studies support the idea that involvement of angular gyrus in number processing is due to a functional relation between number processing and finger representations. There is also supportive behavioral data for this argument. A series of behavioral studies have consistently shown that finger gnosia (finger sense) in younger children is a predictor of numerical abilities (Noel, 2005; Penner-Wilger, et al., 2007). In addition, in our lab we found that finger tapping differentially interferes with finger tapping, showing use of shared resources between addition and finger processing (Soylu & Newman, 2011).

IPS (Intraparietal Sulcus) and the SNARC Effect

IPS is another region that has been consistently found active in a variety of number processing tasks, for example number comparison (Pinel, Piazza, Le Bihan, & Dehaene, 2004) and simple addition (Pesenti, et al., 2000). In the triple-code model it was proposed the IPS, particularly its horizontal segment, is responsible for quantity processing independent from the number notation, and that its function is analogous to one of a “mental number line” (Dehaene, et al., 2003). The mental number line argument is also supported by the SNARC (spatial-numerical association of response codes) effect, which refers to the finding that in a parity judgment test right button responses are faster for large numbers and left button responses are faster for small numbers. This supports the idea that the comparison of numerical quantities takes place on a mental number line extending from left to right. (Dehaene, Bossini, & Giraux, 1993).

However there is evidence challenging the idea of a mental number line for quantity processing. Fischer (2008) explored whether finger-counting habits contribute to the SNARC effect and found that subjects who are left-starters show a SNARC effect significantly more than right-starters. In another study subjects were asked to identify Arabic digits by pressing one of 10 keys with all 10 fingers. The configuration of response buttons varied both in terms of the global direction of the hand-digit mapping and the direction of the finger-digit mapping within each hand, from small to large digits or vice versa. The results showed that subjects performed better when there was a congruency between the reported finger-counting strategy of the subject and the mapping of the response buttons (Di Luca, Grana, Semenza, Seron, & Pesenti, 2006).

Based on the presented evidence it is possible that angular gyrus contributes to a finger based representation of numbers, while IPS contributes to a multimodal representation of numerical quantity. The “mental number line” analogy can still be useful in explaining the function of IPS, while taking into account that the direction and structure of this number line is grounded in bodily dynamics, for example handedness and finger counting habits. In addition the analogy can be modified in a way that

we not only talk about a number line but also hands tracing it during its use.

We need further neuroimaging studies investigating the relation between bodily and basic mathematical processes to clarify the question about the exact roles of angular gyrus and IPS, as well as pre-frontal regions in number processing.

Adopting an Evolutionary Perspective

Since one of the main ideas behind embodiment is the exploitation of simple perceptual and motor neural resources for higher cognitive functions, adopting an evolutionary perspective can help not only in understanding how these functions emerged during evolution but also explaining how they are currently situated in the sensorimotor system. This is also true for mathematics. An evolutionary perspective provides a bigger and more connected picture as to why a distributed network of brain areas is functional in number processing.

One recent theory on the evolution of higher cognition is Anderson’s “massive redeployment theory” (2007). Anderson argues that higher cognition is possible through redeployment of existing neural systems for new functions. By reviewing 135 neuroimaging studies in different domains he provided empirical validation for three predictions: 1) A single brain region is used for many cognitive functions, 2) evolutionarily older brain areas are affiliated with more cognitive functions, and 3) newer cognitive functions utilize more distributed brain areas. Let’s revisit the case of angular gyrus from this perspective. We have already covered how interpretation of angular gyrus activation as verbal processing (Dehaene, et al., 2003) makes it difficult to explain a range of neuropsychological (such as Gerstmann’s syndrome), neuroimaging and behavioral findings. What we currently know about evolution of language can help us in understanding the role of angular gyrus. Arbib (2002, 2005) proposed that human languages followed an evolutionary trajectory including such stages as; the simple grasping movement, understanding actions of another individual, imitation, a manual based communication, and verbal communication, finally yielding to complex human languages. Considering the argument that hand/finger related sensorimotor areas were redeployed for language during evolution, we can expect that verbal processing also use neural resources related to the perception and execution of hand movements. Studies on verb meaning provide support for the proposed relation between the sensorimotor system and language processing. Buccino et al. (2005) showed that action-related sentences modulate relevant parts of the motor system, especially the mirror neuron system. A simulation theory is proposed as one possible explanation for this phenomenon: “. . . the understanding of action-related sentences implies an internal simulation of the actions expressed in the sentences, mediated by the activation of the same motor representation that are involved in their execution” (Buccino, et al., 2005, p. 361).

This partially supports the idea that angular gyrus activation in verbal processing might be due to use of finger processing resources, which is shared by number processing. Although we do not have empirical data to support this claim, the idea here is to show how adopting an evolutionary perspective has the potential to provide alternative explanations that is more consistent with disparate findings on number processing. In this sense, interpretation of data requires consideration of not only the nature of the task, but also its evolutionary past.

Criticisms & Alternative Views

The idea of embodiment of mathematics is not free of criticism. The embodied account of mathematics was particularly criticized by mathematicians who believe that “brain based” mathematics necessarily refuses a “transcendent mathematics”. From this perspective mathematical embodiment negates mathematical realism; that is propositions about embodiment and transcendence mathematics are mutually exclusive. The view that mathematics cannot be a product of the embodied mind since it is transcendent is characterized as “Romance of Mathematics” by Lakoff and Nunez (2000).

However, according to an alternative view the “... fact that human mathematics is based in human cognitive capacities does not mean that these capacities cannot provide recognition of transcendent mathematical truth” (Voorhees, 2004, p. 87). I believe that how we do mathematics and what mathematics is should be studied separately, since the answer to the former question does not inform the latter one. The confusion of these two fundamental questions can blur the study of embodied mathematics by attracting invalid criticism.

A different perspective, which shows that the discussion about the transcendence of mathematics is more philosophical rather than empirical in nature, was proposed by Gödel. He argued that mathematical concepts are as “real” as physical objects: “It seems to me that the assumption of [mathematical] objects is quite as legitimate as the assumption of physical [ones] and there is quite as much reason to believe in their existence” (Longo, 2007, p. 207). However, the mathematical realism of Gödel is not conclusive. He points out that questions that relate to the ontology of physical objects are the same as the ontology of mathematical concepts: “the objective existence of the objects of mathematical intuition ... is an exact replica of the question of the objective existence of the outside world” (Longo, 2007, p. 209). Overall, I believe that studies on mathematical cognition inform how we do mathematics and not what mathematics is.

Conclusion

Mathematics, being one of the most abstract domains of human knowledge, is a challenge to embodiment. There are two types of approaches to the study of mathematical embodiment: 1) Empirical investigation of how bodily processes interact with mathematical processes, and 2) study

of how people use conceptual metaphors to make sense of mathematical concepts; by using already existing knowledge in a physical domain to understand a more complex and abstract mathematical concept. While the former approach provides empirical validation for claims, it does not provide a unified theory of how mathematics is grounded in the sensorimotor systems. The second approach, focusing on the role of conceptual metaphors in mathematical thinking, provides a general theory, but attracts serious criticisms due to lack of empirical validation. I propose that approaching to mathematical cognition as embodied simulation can make it possible to interpret seemingly disparate findings to provide a more comprehensive explanation for how people do math.

I have also reflected on the implications of adopting an embodied and evolutionary perspective in interpreting neuroimaging data. I proposed that a study of the neural underpinnings of mathematical cognition should aim at explaining how the processes studied are grounded in the complex interactions of sensorimotor networks from an evolutionary perspective. Study of the neural underpinnings of mathematical thinking is more about understanding how a complex network of sensorimotor circuitry interact to bring forth mathematical ideas rather than identifying rigidly modularized areas that are only specific to mathematical thinking.

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