

Toward a Mental Decision Logic of the Small-Grand World Problem: Its Decision Structure and Arithmetization

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Abstract

Human decision making is really a two-stage process: the process of forming an appropriate decision problem and then proceeding towards its solution. Often, one needs to work between stages till a decision problem with sufficient information has been constructed. Most current decision theories focus on Stage 2 decision process but neglect Stage 1 decision process (Joyce, 1999); consequently, the so-called “small-grand world” problem (SGW) has remained an open question since Savage (1954). This paper proposes a model of the reasoning processes underlying Stage 1 decision in the form of a mental decision logic (MDL) of the SGW problem, and give an arithmetization of its dynamics through the a novel use of the Gödel numbering method. It explains how MDL works in modeling the SGW problem; the idea is to use domain-specific mental predicate-argument structures (Braine, 1998) in transforming between the act-state structures which are commonplace to most formal theories of decision.

Introduction

*Distinction between Stage 1 and Stage 2 decision processes*When a well defined decision problem is given, which we will explain later, we are facing a stage 2 decision problem. The real pain for the decision-maker may occur during the stage 1 decision process, for during that period the decision-maker might still be trying to formulate the right decision problem. The mind is walking back and forth between a smaller world and a bigger world. There potentially exist a number of mental activities going on in a stage 1 decision process which need to be modeled and taken into account of any decision theory. However, as Joyce (1999) pointed out, most current formal decision theories neglect the stage 1 decision process.

*Organization of this Report*Due to the highly interdisciplinary nature of the author’s research interests, this document contains a substantial amount of background material. I will endeavor to present it clearly and concisely, providing only the essentials for understanding MDL, as I have described it herein. Section 2 will cover normative concepts in the decision-

theoretic literature, including a formal description of the SGW problem. Section 3 will serve to give a concise introduction to the psychology of reasoning, with a heavy focus on *mental predicate logic*, as it has been developed by a number of researchers. In section 4, we begin to develop *mental decision logic* to describe the cognitive/mental dynamics of small and grand-world interaction. . Finally, in section 5, we give an arithmetization of these dynamics through the exploitation of various number-theoretic properties, and a novel usage of the Gödel numbering method.

Decision Theories and the SGW Problem

A number of proposals for theories of decision have been made by a variety of researchers throughout the years. I present a very brief list of some of the most influential of these theories:

- Von Neumann and Morgenstern (1944/1990): A decision problem has a two-layer structure: Choices and outcomes. Each choice is associated with a number of outcomes; thus a choice itself may serve as an event or may be naturally assigned an event. In a two layer structure, both desirability and feasibility are associated with an outcome.
- Savage (1954/1972): A decision problem has three layers: action functions, states as possible descriptions of the world, and an (dis-positioned) outcome space. The three-layer structure has some advantage in that it can separate desirability (associated with outcomes) from feasibility (associated states). This three-layer structure requires a fourth component, the partition function in order to yield a set of events. We will explain why such a structure causes some two-folded difficulty in modeling the small-grand world problem (SGW). Note, this is a very valuable problem in nature. The SGW problem is not only a problem in modeling, but a problem in human decision-making so it has to be taken into account.
- Jeffrey (1965/1983): Collapses all aspects of a decision problem into propositional form. Jeffrey is considered the father of the evidential tradition in decision

theory. Every component is transformed into a propositional description, allowing logic connectives, and enabled Ethan Bolker (1966) to establish a cleaner representational theorem. Jeffery's (propositional) logic of decision seemingly avoided the SGW problem; but to my view, it merely lost it through cutting down on richness in representation..

- Luce & Krantz, (1971): Developed an event-conditional approach in the direction that was getting closer to what I deem to be psychological plausibility. However, new questions have been pointed out by Joyce, concerning the strength of the event-conditional treatment, and whether it loses the small/grand worlds distinction.
- Joyce (1999): Propositional, four-layer (4-components) approach in causal decision theory. It is also event conditional to claim a unified account for both evidential and causal theories. Joyce's account is in the right direction because it is propositional and event-conditional, and it has kept the SGW problem open.

Grand-World Decisions (Syntax consistent with Joyce 1999) A decision problem can be defined as consisting of four components: $\mathbf{D} = (\Omega, F, S, O)$. Here S is a set of states; each state can be seen as a possible description of the world. F is a set of action functions such that $F: S \rightarrow O$, where O is a set of outcomes. For any $f \in F$ and any $s \in S$, $f(s) = O$ (f, s) is an outcome. Sometimes, we also call $f(s)$ is an outcome, $O(f, s)$ being dis-positioned. Ω is a partition function. E.g., $\Omega(S) = S'$ is a partition of S ; i.e., each element in S' is a non-empty subset of S , called an event. The events in S' are mutually exclusive and collectively exhaustive to S . In symbolic, for any s'_i and $s'_j \in S'$, $s'_i \cap s'_j = \emptyset$, and for all $s'_i \in S'$, $\cup s'_i = S$.

The syntax of an axiomatic decision system naturally includes a set of axioms. Some axioms are instrumental, which promise what kinds of action functions may be admitted (e.g., constant action in Savage's system). Some axioms are about preference ordering (e.g., non-triviality axiom preserves the existence of partial ordering). It involves a great deal of theoretical issues about axiomatization, which is beyond the concern of our current discussion. What is important here is that Savage's system is designed for what he calls the grand-world decision or decision in some isolated situation. It assumes that the agent would take all the possible options into account and could evolve the partition of the states to the highest level of pertinent detail. Joyce provides a general definition of the grand-world decision problem below, by using a mixed language of Savage and Jeffrey

$\mathbf{D}^G = (\Omega^G, F^G, S^G, O^G)$ is the grand-world decision problem that an agent faces if and only if there is no proposition X , whether is in Ω or not, such that the agent

strictly prefers $(O \& X)$ to $(O \& \neg X)$ for some outcome $O \in O^G$.

In other words, \mathbf{D}^G is the decision problem whose outcomes function as unalloyed goods relative not only to the propositions in Ω , but to all the propositions that there are. When a decision problem fails this test it is a small-world decision.

Small-World Decisions As Savage acknowledged, his system about grand-world decision is an idealization, which can hardly be realistic in human decision-making. He wrote that it is difficult to say with any completeness how such isolated situations are actually arrived at and justified (1972, p83). By analyzing the "Jones" decision example in his *Foundations of Statistics*, Savage suggested the term, "small world" decision problem as a microcosm of the grand-world situation. Note that switching attention to focusing on modeling small world decision problem, as my psychologist colleagues might consider, without considering grand world situation might not help here, as the SGW problem is actually bi-directional. Savage seemed more interested in describing how move from the grand world to a small world situation, while Joyce concerned himself with movement in the other direction. Joyce's (1999) description is perhaps the clearest I could find.

Every small world decision \mathbf{D} is a coarsening of the grand world problem \mathbf{D}^G , and there is always a sequence of refinements $\mathbf{D}, \mathbf{D}_1, \mathbf{D}_2, \dots, \mathbf{D}^G$ that begins with \mathbf{D} and ends with \mathbf{D}^G . Choosing is really a two stage process in which the agent first refines her view of the decision situation by thinking more carefully about her options and the world's states until she settles on the "right" problem to solve and then endeavors to select the best available course of action by reflecting on her beliefs and desires in the context of this problem. Normative decision theories have concentrated almost exclusively on the second stage of this process. Once the decision problem is in place they try to explain what makes the choice of an action rational or irrational. At this point, to the author, a behavioral or psychological model might not do any better if instead of concentrating on the grand world problem, it concentrates on a small-world problem only. The initial stage is equally important, however, and any complete account of human decision-making must have something to say about it. A formal model of the refinement process by Joyce can be briefly described as follows. Suppose one decision $\mathbf{D}^+ = (\Omega^+, O^+, S^+, F^+)$ is a refinement of another $\mathbf{D} = (\Omega, O, S, F)$ just in case O^+ is a refinement of O , S^+ is a refinement of S , and F^+ is a refinement of F . (It follows that Ω must be a subalgebra of Ω^+). Note that psychologically, it is equally interesting how people move from a refined \mathbf{D}^+ decision situation to

a less refined decision, due to limited accessibility, limited working memory, or mental model construction. People may even go this direction purposely as part of their decision efforts in order to reduce their cognitive workload. This follows Wittgenstein's view (1969/1972) that subjective certainty should play an important role in decision processes dealing with uncertainties. This is probably even closer to the truth in the information (over-loading) age.

As Joyce pointed out, "the rationality issues concerning the SGW problem is (a) some explanation of what it takes for a small-world decision maker's estimates of her grand-world attitudes to be correct, and (b) an account of rationality that applies to both grand- and small-world decision making and that guarantees that any small-world decision maker who correctly estimates her grand-world attitudes, and who adheres to the law of rationality, will make a small-world choice that rational when viewed from either the grand-world or small-world perspective." (1999, p77) The rationality discussions would have to do with the utilitarian decision semantics, which is beyond the scope of present paper.

Structural Puzzles Concerning the SGW Problem

Puzzle 1.

By Savage, "The small-world states are in fact events in the grand world, that indeed they constitute a partition of the grand world." (1972, p84) In the technical footnote on the same page, he even suggested not to insist that the small world have states at all, but rather to speak of a special class of events as small-world events. Let S be the set of the grand world states. The construction of a small world S' from the grand world S begins with the partition of S into subsets, or small world states. The puzzle here is that with or without full knowledge about all the grand world states, the selected events in a psychologically plausible small world need not be either mutually exclusive or collectively exhaustive. In other words, given S , a small world S' does not have to be a partition of S . In next section, when we formulate mental decision logic, it will suggest that we replace the partition function Ω by n-place predicates, of which Ω can be treated as some special cases, but special cases only.

Puzzle 2.

Savage proposes two principles below to specify the relation between the structures of grand-world decision and small world decision.

Principle 1. A small world consequence is a grand world action. Let f' be a small-world act function, and s' a small-world state or a grand world event (i.e., $s' \in S'$ and $s' \subseteq S$). Assume that f is a grand-world act function, which can be defined as;

$$f(s) =_{\text{Df}} f'(s \mid s \in s')$$

In this sense, he also has:

Principle 2. Each small world act function raises a grand world act.

Together two principles are a bit confusing and need some clarification. By Principle 1, a small-world consequence (outcome) is based on only one small-world state, which is a grand world event that is only a subset of S . But a grand world act function should be defined on S but not in S' . But by Principle 2, a small world act should raise a grand world function. My understanding is that what Savage means must be that a small world function f' is defined on S' , taken each s' in S' to result in a consequence. Thus, a grand world act has to be defined by the set of small-world consequences yielded from a small world act function. I consulted with Joyce about my interpretation, and he agreed. (Personal communication, October, 2004). This clarification has proved to be very helpful. First, in Section 5, when we try to formulate a mental predicate decision logic, it will require three layers of individual variable: variables that range over S , variables ranging over S' , and variable ranging over an event. Second, if a small world consequence is a formula, $f(s')$, a grand world act can then be represented as a set of formulas. As an analogy, this is parallel to a statement and a proof logic, respectively. In Section 6, when we try to do Gödel number coding, it will allow us to elegantly code a small world outcome as an expression, and a grand world act as a sequence of expressions.

Puzzle 3.

From psychological point of view, the current decision theories lack a mental reasoning mechanism that would allow a decision maker to work back and forth between the grand-world decision and a small world decision problem. Such a mechanism must be bi-directional. In Section 4, we will work toward a mental decision logic to fill this gap. The main idea is to allow bi-directional transformations between small-world act-event structures and grand-world act-state structures through mental predicate-argument structures.

Puzzle 4.

In Savage's decision structure, as well as in Jeffrey's logic of decision and Joyce's causal decision theory, the partition function Ω is important because it guarantee the system to satisfy the requirements of a Boolean algebra. Now, as explained in Puzzle 1 above, we are going replace partition functions by n-place predicates, we need to work out the algebra structure of the resulting mental predicate decision logic of the SGW problem. In Section 5, we first provides an arithmetization by applying Gödel number method, then show that the resulting algebra

structure is a ring on integers, with even numbers as its ideal.

Mental Predicate Logic

Mental predicate logic One thing that sets up mental model theory and mental logic theory as major *competing* approaches in psychology of reasoning is that each has its mental representational systems, from which predictions can be made. Most researchers, including Braine and Johnson-Laird, view the mental model representations as purely semantic, and mental logic representations (i.e., inference schemas) as purely syntactic. From the viewpoint of mental metalogic (Yang & Bringsjord, 2001, 2003, and forthcoming) this is a false dichotomy that stands in the way of progress toward greater understanding of human reasoning. For example, in case of Braine's mental predicate logic system, the quantified version of Modus Ponens is formulated as follows:

For all the $x \in X$, $A(x)$. Therefore, for all the $y \in Y$, $A(y)$, where $Y \subseteq X$.

Here both domains X and Y are bounded by the definite particle "the", which can function as a universal quantifier. Note that the individual domain is a semantic component in standard value-assignment semantics of first order logic. But in mental predicate logic, this semantic component is construed into the form of an inference schema; though mental logicians used to claim no need for a semantics in mental logic theory.

Mental Decision Logic

Mental Decision Logic Here, by a mental decision logic it does not mean some standard logic system, nor a complete mental decision logic accounting for any full decision theory, which will require a great deal of further research. As a starting point, the initiation of mental decision logic for the SGW problem given below aims to provide a formal language that can represent the SGW in a mental predicate logic format compatible with the formal language used in 3.1; by doing so, it will allow to apply mental predicate logic mechanisms sampled above to model the SGW problem.

In the following a list of lexicons for mental decision logic will be given with necessary explanations concerning mental logic. Let $S = \{s_1, s_2, \dots\}$ be the set of all the grand world states, write the power set of S as $P(S) = \{s' | s' \subseteq S\}$, and denote a possible subset of $P(S)$ by S' .

Two kinds of constants are needed.

(a) A set of individual state constants: a_1, a_2, \dots . Each state constant a_i can be used to name a grand world state $s \in S$.

(b) Another set of individual event constants: b_1, b_2, \dots . Each event constant b_i can be used to name an event $s' \subseteq S$.

Three kinds of state variables are needed.

(c) The grand state variables: x^1, x^2, \dots , with infinite supply, each x^n ranges over S .

(d) Event variables: x_1, x_2, \dots ; each x_i ranges over a particular set of possible events,

which in general are not necessarily either mutually exclusive or collectively exhaustive; but an any given partition of S can be treated as a special case.

(e) x_i^j ranges over the j^{th} event ($s' \subseteq S$) in the domain of x_i ($i, j = 1, 2, \dots$) when x_i is given; otherwise, each x_i^j is being held as a frame for later assignment to range over the states in some possible event, or say, a subset of S .

Only one kind of predicates is needed.

(f) A_k^n are a n -place mental predicates, $k = 1, 2, \dots$

Note that for items (c)-(f) above: As a psychological model, Braine's mental predicate logic has no formal semantics. An individual variable alone is not assigned to any individual domain. Similarly, a predicate alone is not committed to any truth condition. When and only when a predicate-argument structure is formed, it will be, and has to be associated with some specified individual domain. For example, $A_k^1(x)$ specifies S as its domain; $A_k^1(x_i)$ specifies some S' as its domain; and $A_k^j(x_i^1, \dots, x_i^j)$ specifies j events in S' as a set of multiple domains (i.e., each x_i^j is assigned an event that is a subset of S as its domain).

Also note that for items (g) and (h) below: By current decision theories, act functions are always monadic. Here we will still keep n -place function symbols for two considerations. One is that this treatment may allow other non-logical functions to be construed into the system later when we have to deal with utility functions, which are decision-semantic. Another is that it would leave room for potential development of n -place act functions later when we can make some interesting sense of them. However, in modeling SGW problem, it is convenient and makes sense to make a primary distinction at the atomic level between small world acts and grand world acts. In the following, letter f is used to denote a grand world function and h a small world function (usually letter g is used for a second function, but we try to avoid possible confusions as we will use g for Gödel number function in next section). Attention: please keep in mind that we DO NOT want to make such a distinction among predicates because the whole point for mental decision logic is to use mental

predicates to manage the switching back and forth between small- and grand- world decision components.

Two kinds of act function are needed:

- (g) f_k^n denote grand-world act function ($k, n > 0$)
- (h) h_k^n denote small-world act function ($k, n > 0$)

Thus, for example, $f_k^1(x)$, $f_k^3(x^1, x^2, x^3)$, or $f_k^1(x_{i1})$ should be well-formed formulas, and $h_k^n(x_i)$ should be well-formed formulas in mental decision logic, while $h_k(x^i)$ and $f_k(x_i)$ would not. It is easy to set up formation rules for mental decision logic. Given the why of formatting the SGW into a logic language as specified above, it is also not too hard to see how mental logic theory sampled in 3.1 may apply. However, to go beyond that, it would demand certain efforts to formulate other decision components. These duties are beyond the call of the present paper.

Processing Program Though to fill out details showing how mental decision logic would work require much follow up research, what we have done so far enables us to outline how mental decision logic functioning in Stage 1 decision processes.

First, suppose the decision maker is trying to move from the grand-world decision problem to a small world decision problem. The following steps should be passed:

Step 1. Looking the grand-world states set S ; he doesn't have know the whole S .

Step 2. Give the current propositional attitudes (beliefs, concerns, interests, et. al.), to frame the content of a predicate A_k .

Step 3. Choose an n -place predicate A_k^n by clustering interested states in S (not necessary all the states in S) into n groups. In other words, a set of subset S' is selected.

Step 4. (This step could be very implicit and may or may not occur). Assign local state variable x^i to each of the n groups. At this point, an n -place predicate-argument structure $A_k^n(x_i^1, \dots, x_i^n)$ has been formed and n sub-domains s' specified. At this stage, conceptually each s' is an event as a subset of the grand-world S.

Step 5. For each s' has been specified, disregard the grand-world statue of any s in s' , and treat s' as a solid single entity. Conceptually, at this point, a grand world event has been transformed to become a small-world state. Cognitively, this is import in mental processing, because this conceptual transformation may cost certain deliberation efforts. It is not hard to speculate that this transformation process can only be done in an event by event fashion. Even after being clustered into a group,

each grand-world state might still carry some significant different grand-world features. (The literature in psychology of categorization and conception would have a lot to say about this.)

Step 6. Now the S' selected in Step 3 is not longer treated as a collection of groups consisting of grand-world states, but conceptually become the set of some small-world states. Thus, this S' provides the decision environment necessary to resist a small world decision problem. In formulating a decision problem in this small world, one needs to delete (yes, one need to delete, and deleting might have some deliberation cost) those local grand-world-state variables (x_i^1, \dots, x_i^n) used earlier, and replace them by initiating a new, so-called event variable x_i that ranges over S' , which is now the set of small-world states.

Step 7. Here we have S' and an event variable x_i , together they can call for different monadic predicates A_k^1 , and generate different predicate-argument structures $A_k^1(x_i)$.

Step 8. In turn, each resulting predicate-argument structure $A_k^1(x_i)$, $x_i \in S'$, initiates a small world act function h_k^1 . Each $h_k^1(x_i)$ will return a local set of small world outcomes (consequences), and for a given s' in S' , $h_k^1(s')$ is treated as a small-world outcome.

Till Step 8, we have moved from a grand-world decision problem to a small-world decision problem. And after Step 8 Stage 1 decision process ends and Stage 2 decision process starts. Then any current decision theory can step in, and take over to tell either an evidential, or a causal, or an integrated decision theory, depends on what kind of utility account it associated with.

Arithmetization of MDL

Why arithmetization? There are two reasons behind providing an arithmetization of mental decision logic of the SGW problem, both concerning to keep the standards common to modern normative decision theories. One reason is that in the normative decision structure, Ω is a partition function the guarantees the resulting decision structure as a Boolean algebra. In mental decision logic of the SGW problem, we withdraw this partition requirement, and replace Ω by mental predicates. By taking this approach, the decision structure can still be closed under the set-theoretic union function, but is no longer closed under relative complement. (Consider that now it is possible that $\cup B \neq B$.) Though the resulting decision structure doesn't have to be a Boolean algebra, we do need to see what algebra structure the resulting decision logic commits to.

Gödel numbering. Given that it contained non-logical decision-functions, the mental decision logic of the SGW problem tends to be a first-order theory, not first-order logic. There are many ways to do Gödel coding. The method we use below is a modified version of Mendelson (1979). For the mental decision-logic (MDL) described in Section 5, we correlate with each symbol u of MDL a positive integer $g(u)$, called the Gödel number of u , in the following ways.

$$g(()) = 3; g() = 5; g(,) = 7; g(\neg) = 9; g(\rightarrow) = 11$$

$$\begin{aligned} g(a_k) &= 5 + 8k \text{ for } k=1, 2, \dots \\ g(b_k) &= 7 + 8k \text{ for } k=1, 2, \dots \\ g(x^n) &= 11 + 8n \text{ for } n = 1, 2, \dots \\ g(x_k) &= 13 + 8k \text{ for } k = 1, 2, \dots \\ g(x_k^n) &= 17 + 8(2^n 3^k) \text{ for } k \geq 1 \\ g(A_k^n) &= 19 + 8(2^n 3^k) \text{ for } k \geq 1 \\ g(f_k^n) &= 23 + 8(2^n 3^k) \text{ for } k \geq 1 \\ g(h_k^n) &= 29 + 8(2^n 3^k) \text{ for } k \geq 1 \end{aligned}$$

Given an expression $u_1 u_2 \dots u_r$, we define its Gödel number to be

$$g(u_1 u_2 \dots u_r) = 2^{g(u_1)} 3^{g(u_2)} \dots p_{r-1}^{g(u_{r-1})} p_r^{g(u_r)},$$

where p_i for the i^{th} prime, and $p_0 = 2$.

For example, an expression in mental decision logic of SGW can be small-world outcomes $h_k^{-1}(x_i)$, in which u_1 is h_k^{-1} , u_2 is “(”, u_3 is x_i , and u_4 is “)”.

For an arbitrary finite sequence of expressions $e_1 e_2 \dots e_r$, we can assign a Gödel number by setting

$$g(e_1 e_2 \dots e_r) = 2^{g(e_1)} 3^{g(e_2)} \dots p_{r-1}^{g(e_{r-1})} p_r^{g(e_r)},$$

where p_i for the i^{th} prime, and $p_0 = 2$.

Interestingly, as we discussed earlier, in mental decision logic of SGW problem, a grand world act can be given as a sequence of the small-world outcomes yielded from the same small-world act function. For example, an f_k can be defined by $h_k(x_i^1), \dots, h_k(x_i^j)$.

Thus, each symbol is assigned a unique odd number; each expression is assigned an even number and the exponent of 2 in its prime factorization is odd; while each sequence of expressions is assigned an even number and the exponent of 2 in its prime factorization is even. In other words, g is a one-one function from the set of symbols, expressions, and finite sequences of expressions of MDL into the set of positive integers. The power of Gödel number method is that by the uniqueness of factorization of integers into primes, a Gödel number can be uniquely decomposed to its factorization, and thus to recover the original expression or the sequence of

expressions. (Further discussions are beyond the scope of this paper)

A ring/ideal structure Gödel number method, as it is well known, is one of the main techniques created by Gödel in proving his completeness theorem of first-order theory (1931). And this method has developed to study algebra structures of formal systems in model theory, a branch of mathematical logic. It is also called number-theoretic semantics in model theory (see Mendelson, 1979). Below is the definition of a specific algebra structure (Gratzer, 1968/1979).

Definition Let R be a *ring*, and I is a subset of R . I is an *Ideal* of R if for any $a, b \in I$, $(a - b) \in I$, and for any $a \in I$ and any $r \in R$, $a \times r \in I$.

By the definition, the set of all the integers is a ring, and its subset of all the even number is an Ideal. (Gratzer, 1979)

Note that in 6.2., all the MDL symbols are coded by integers, of which small-world outcomes that represented as expressions and grand-world outcomes that can be represented as sequences are assigned to even numbers. Thus we treat the arithmetization of the MDL of SGW problem as a ring, the Gödel numbers of outcomes can be seen as its ideal. This method is also used in model theory, and is also called number-theoretic model of first-order theory.

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