The Role of Analogy in Teaching Middle-School Mathematics

Lindsey K. Engle (Lengle@psych.ucla.edu)
Department of Psychology, University of California, Los Angeles
Los Angeles, CA 90095-1563

Keith J. Holyoak (Holyoak@lifesci.ucla.edu)
Department of Psychology, University of California, Los Angeles
Los Angeles, CA 90095-1563

James W. Stigler (Stigler@psych.ucla.edu)
Department of Psychology, University of California, Los Angeles
Los Angeles, CA 90095-1563

Abstract

Analogies produced in twenty-five US eighth-grade mathematics classroom lessons were analyzed according to their frequency and structure. Frequency findings suggest that analogies are a common part of mathematics classroom learning, and a component analysis revealed regular structural patterns in the way these analogies are produced. Teachers tended to organize the analogies by producing the target, source, and mapping steps before students become active participants. Students were most likely to then make inferences, adapt them to the target context, and solve target problems. Student participation was either independent or co-constructed with a teacher or other students. Findings address an important correlate with experimental research on analogical reasoning.

The United States' educational system is presently struggling to find teaching programs that facilitate mathematical understanding that goes beyond algorithmic knowledge. Standardized test results recently released in California indicate that despite improvements, the educational system remains far below state goals in mathematics (STAR, 2001). One major component of this difficulty is a lack of knowledge about how to teach abstract concepts so that students are able to transfer this learning across contexts.

Systematic use of analogy may be integral to a teaching program that meets that goal. Analogy is a comparative structure that highlights abstract structural relations (Gentner, 1983), and facilitates schema acquisition and transfer across problems (Gick & Holyoak, 1983). Learning mathematics requires development of generalized concept representations that can be applied across contexts (Bransford, Brown & Cocking, 1999).

Cognitive scientists have argued for decades that analogy plays a central role in human cognition, learning and problem solving (e.g., Holyoak, Gentner & Kokinov, 2001; Kolodner, 1997; Holyoak & Thagard, 1995; Gentner & Toupin, 1986; Piaget, 1950). However, there have been paradoxical findings in analogy research. While analogy has been demonstrated to be used in several everyday contexts (e.g. Dunbar 1995, 2001), most laboratory studies show low rates of spontaneous noticing and use of analogies for problem solving (e.g. Gick & Holyoak, 1980, 1983). It is necessary to understand this discrepancy between observed patterns of analogical reasoning in the laboratory and in everyday contexts, termed the analogical paradox (Dunbar, 2001), in order to design meaningful interventions to promote educational usage of analogy.

We suggest that in order to clarify the paradoxical findings concerning analogy use, detailed analysis of everyday analogy usage is essential, because important aspects of analogy use can only be understood through online analysis of the pragmatics governing analogy production. The current study uses discourse analytic techniques to explore analogy production in the context of teaching mathematics in eighth-grade mathematics classrooms.

Methods

Sample and Coding

Twenty-five videotaped eighth-grade mathematics lessons were analyzed to examine analogy activities. The lessons were randomly selected from a larger random probability sample collected as part of the Third International Mathematics and Science Study (TIMSS) directed by Jim Stigler (see Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1998). All selected classrooms were videotaped on one occasion. The classrooms were selected from US public, private, and parochial? schools in both urban and rural areas, and videotaping was conducted throughout the school year. The lesson content was not constrained, but most lessons drew from number theory, geometry, or algebra domains. Teaching styles similarly were not constrained and thus reflected a range of techniques and perspectives.

Lessons were analyzed using V-Prism, a computer software package designed to allow simultaneous viewing of a digitized video and its typed transcript on a computer screen. In the program, the video's transcript is time-linked so that the lines of text move temporally with the video.

The transcript may be marked with codes to designate when an episode begins and ends.

Six levels of analysis were developed such that every lesson was coded in six passes. Each pass was conducted by at least two coders and intercoder reliability was assessed. Codes and frequency data are presented below.

1. Identifying analogies

The definition of analogy used in this study was based on Gentner's structure mapping model (1983). Analogy was operationalized as a comparative structure between familiar objects, termed the *source* (or *base*) of the analogy, and relatively unfamiliar objects, termed the *target* of the analogy. *Objects* were defined as entities that function as wholes at a given level of analysis. The source and target objects are aligned according to their *predicate*, or relational, structure such that inferences are drawn from the source predicate structure to explain the target predicate structure. *Mapping* is defined as the process of aligning and drawing inferences between the source and target objects. Inferences are then drawn from the source structure and used to derive novel knowledge about the parallel target structure. Several constraints govern which mappings are made in each analogy, since all possible mappings are not typically completed (see Holyoak & Thagard, 1989).

Coders used a conservative measure of analogy, marking only units where a source, a target, and clear structural mapping between the source and target could be identified in discourse or explicit gestures. Two examples illustrate typical analogies. Reliability between the two coders was calculated on 105 protocols in 4 lessons (18% percent of the total sample). Agreement was 86%. Differences were resolved in discussion and consensus between the coders.

Figure 1 provides an illustration of the type of analogy typically identified in the data. This is a transcript of an analogy presented by a teacher to a whole class. The teacher is standing at the board, where the formula for circumference and a drawing of a circle is projected behind him.

Construction of Source object: *Highlighting the predicate structure:* *External layer of (peel, orange)*	`Teacher: Now here's how I alway's looked at it. We're gonna say this- this circle right here is an orange Its an orange Alright? Its an orange Now lets say we're gonna take - stick a needle in the orange n' suck out everything inside except for the peeling of the orange. ((demonstrates gesturally)).`
Mapping: orange peel to circumference	`Okay we're gos- we're gonna pretend like that's our circumference right there.`
Adapting the inference (this object will be an external layer) to the context of a geometric circle	`((teacher uses a pointer to run along the outside edge of the circle on the overhead projector))`

Figure 1: Example Analogy

A total of 103 analogies were identified as verbally produced by classroom participants. Lesson included a mean of 4.1 analogies ($SD = 2.6$). The range for individual lessons was between 1 and 11 analogies. Thus in every lesson examined, at least one explicit analogy unit was coded.

2. Participant Organization.

Each analogy unit was examined to record the roles played by teachers and students. Based on empirical and computational models of analogy, coders specifically determined whether a teacher, a student, a group of students, or no one generated the four steps of analogical reasoning demonstrated above in figure 1 (source, target, mapping, inference and adaptation, and problem solving).

Findings revealed that most analogies were initiated by teachers, and those produced by students tended to be highly superficial and only occurred in two classrooms (3% of the total number of analogies coded). Teachers produced 84% of sources, 77% of targets, and 89% of mappings. Students had a comparatively more active role in developing inferences and adapting them to target contexts, produced 27% alone or collaboratively with the teacher. Finally, students were most active in completing computations to provide final answers to target problems. Although only 42% of analogies had an explicit answer, students supplied 38% of those answers.

3. Source Structure

In order to examine the types of sources and targets produced typically in these mathematics analogies, the structure of the source and the target were independently coded for each analogy unit. Five classification categories were used. There were: *1)* decontextualized math problem, *2)* contextualized math problem (i.e., a word problem set in a non-math context), *3)* schema (general rule, no surface features), and *4)* outside math phenomena. If more than one source was used to explain the same target, or the target was stated in two ways, the source was coded " multiple." See Figure 2 for sample analogies of each structure.

Intercoder reliability was calculated separately for coding source and target. 22 protocols were used from 3 lessons (approximately 15% of the analogies coded). Reliability was calculated to control for chance using Cohen's Kappa, yielding k=72 and k=81 for source and target reliability (acceptable to good levels).

Both the sources and targets displayed significant differences between the frequency with which each structure category was used, χ^2 (4) = 26.5; 68.4; p <.001 in both cases. While both sources and targets were most likely to be decontextualized math problems (40% and 44%), all four categories were represented. The next largest proportion of targets were math schemas (33%), suggesting that teachers were using analogies not only to prompt solutions to single problems, but also to aid in developing more general schemas. The most substantial distinction occurred between objects outside the domain of mathematics used as sources and targets. 15% of sources were outside the domain of mathematics, while only 1 out of 103 targets was a non-math phenomenon. This is not entirely unexpected since the math classroom is oriented towards mathematics learning, but it is conceivable that mathematics classrooms would discuss how to apply math structures to understand real-world problems. Finally, there were more multiple sources used (16%) than multiple versions of the target (5%).

Table 1: Structural composition of analogy sources and targets

STRUCTURE	FREQUENCY	
	Source	**Target**
Not math	15	1
Contextualized	19	18
Decontextualized	41	45
Schema	12	34
Multiple	16	5
Total	103	103

The overall level of similarity between the surface features of the source and the target was coded on a four point scale, as suggested by research indicating that this relationship governs the reasoning used to solve an analogy (e.g. Ross, 1987; Gick & Holyoak, 1983).

	Source	**Target**	**Surface Similarity**
1	Outside-math phenomena: *It s like balancing a scale, matter doesn t disappear, so to keep it balanced, whatever we do to undo one side we have to do to the other.*	Decontextualized math problem: *It is divided by negative sixty, so we multiply by negative sixty on both sides.*	Far distance
2	Contextualized math problem: *Lets say you ve got money. If you lost 88 cents and then you lost 5 cents more, would you add or subtract to find out the total amount you lost?*	Math schema *When you have a negative number minus another number, do you add or subtract?*	Schema involved
3	Decontextualized math problem: *Ok, don t put all that other stuff. What if it was just 16/20. How would you reduce it?*	Decontextualized math problem: *Now lets change the integers to monomials with variables. So then how would we reduce* $15xy^2z^4/25x^3y$*?*	Low similarity
4	Decontextualized math problem: *How would you multiply these?* (x + 2y)(x + y)	Decontextualized math problem: *In that case, how would you multiply* (5x+ y)(x + 3y)	High similarity

Figure 2: Analogy structure and surface similarity

Table 2 shows the frequencies of the surface similarity between analogy sources and targets. There were no significant frequency difference among the four categories, χ^2 (3) = 3.058 p=.3, indicating that all four types of analogy constructions were regularly used.

Table 2: Surface similarity between source and target

SURFACE SIMILARITY	FREQUENCY
Schema involved	28
Far distance// outside-math source/target involved	19
Low surface similarity	31
High surface similarity	25
Total	103

4. Function

The function of each analogy unit was coded to examine the context and purpose for analogy use. Primarily, this code examined whether the analogy was directed toward explaining a concept, a procedure, or a combination of the two. Additionally, a separate code marked whether the analogy was implemented following evidence of a student having trouble with a problem or concept.

	Mathematical Function	Paraphrased Examples
Code 1	Being a math student.	*Remove the parentheses, very carefully. Kind of like if you were a bomb squad called in to diffuse a bomb. If you mess up the first step, the whole problem will blow up in your face at the end!*
Code 2	Teach concepts only.	*When you're adding fractions, think about your denominators as units, like centimeters or feet. When you add length, the units must be the same in order to add them.*
Code 3	Teach concepts and procedures	*Ok do you remember how we found the perimeter and area of polygons last week? This time it is the same concept but we are going to use the formulas to solve for circumference and area of a circle.*
Code 4	Teach procedures only.	*What were the steps we used to solve the last example? OK lets do the same thing in this problem. First lets factor the numerator and denominator and then we'll see what we can cancel..*

Figure 3: Analogy Function

The mathematical function of each identified analogy revealed that some functions were more frequent than others χ^2 (3) = 20.1, p<.001. The raw frequencies are shown in Table 3. Teachers were most likely to use analogies to teach procedures only. The frequency distribution may be related to the tendency of US teachers to teach more procedures than concepts during a single lesson (Stigler et. al, 1998).

Table 3: Frequency of mathematical analogy functions.

FUNCTION	FREQUENCY
Being a math student	12
Concepts only	19
Concepts and Procedures	30
Procedures only	42
Total	103

We were interested in whether the function or purpose of the analogy was related to the type of analogy produced. Two steps were taken to investigate this question. First, a Pearson chi square was performed to compare analogy function with the structure of source generated. This test revealed that there is a significant relationship between the source generated and the structure of each analogy, χ^2 (12) = 35.8, p<.001.

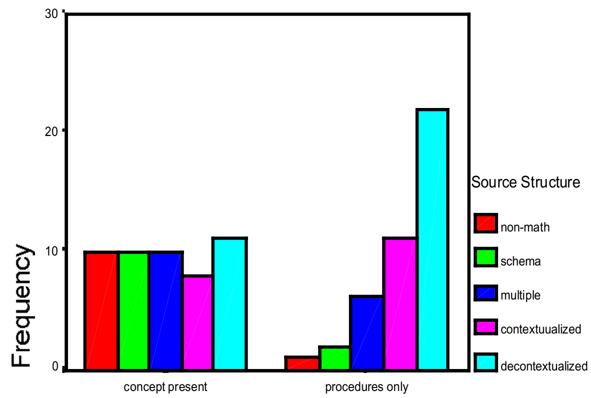

Figure 4: Source structure according to math teaching function.

Second, in order to determine whether the structure of the whole analogy is related to the function, a Pearson chi square was performed to compare the surface similarity between the source and target of each analogy with its function. This test revealed that there was a significant relationship between these variables, χ^2 (3) = 37.5 p<.0001. Far distance analogies were almost exclusively used to explain concepts or concepts paired with procedures. In contrast, relational mappings with high surface similarity between sources and targets were almost exclusively used to teach procedures only. Schemas were more likely to be used in analogies demonstrating concepts, but were also regularly used to teach procedures. Low surface similarity analogies, and mappings based on a decontextualized mathematics object, were used more frequently to teach procedures than concepts.

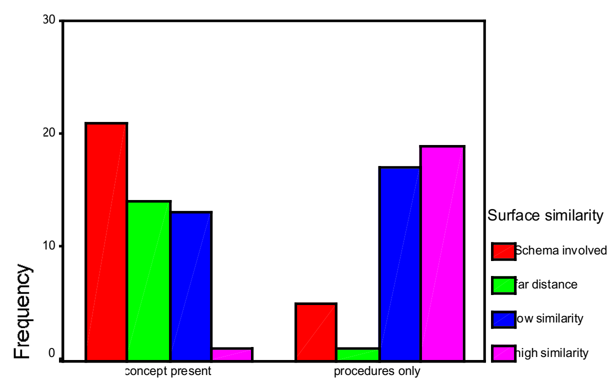

Figure 5: Surface similarity according to math teaching function.

Discussion

The present findings suggest that analogies are regularly produced in US mathematics classrooms, and are generated in reliable structural patterns. These patterns vary as a function of the immediate context and purpose of the analogy. The specific nature of these relationships suggest that analogy generation is highly constrained and organized by the goal of the analogy. When the goal of the analogy was to teach conceptual information, teachers were more likely to use distance analogies, schemas, and lower surface similarity than when they were teaching math procedures alone.

Patterns of analogy usage also revealed interesting associations between teachers' analogy practices and previous research. Teachers frequently presented multiple sources, techniques that have been demonstrated to facilitate schema acquisition and transfer (e.g., Gick & Holyoak, 1983). Thus it is possible that teachers may be using intuitive theories of analogy to guide their analogy production towards an effective teaching tool.

In addition, teachers provided substantial structure for each analogy, perhaps because they are aware that noticing analogies is a difficult task, as has been frequently demonstrated experimentally. While the teachers assistance makes the analogy more likely to be completed successfully than if students were responsible for generating more components, however, this design provides little information to the teacher about whether students are actually performing analogical reasoning when analogies are produced in this way. Although teachers frequently asked students to participate in producing analogies, they may not be aware that the way most students participate does not require that they perform higher order comparative reasoning.

Further, educational research suggests that enabling students to generate predictions and inferences about unknown problem types can be highly beneficial for their transfer performance (e.g. Carpenter, Fennema, Fuson & Hiebert, Human, Murray, Olivier & Werne, 1999). These findings therefore suggest that teachers might be reducing the learning benefits of analogy by highly structuring the comparisons for students. This is an empirical question that is currently under investigation.

Acknowledgments

Development of the TIMSS database was funded by the National Center for Education Statistics (NCES), the US Department of Education, and the National Science Foundation (NSF) grants to James Stigler. This research was additionally supported by an National Institute of Mental Health (NIMH) Developmental Cognitive Science Training Grant to Lindsey Engle.

References

Bassok, M. (2001). Semantic alignments in mathematical word problems. In D. Gentner, K. J. Holyoak, & B. N. Kokinov (Eds.) *The analogical mind: Perspectives from cognitive science* (pp. 401-433). Cambridge, MA: MIT Press.

Brown, A. L., & Kane, M. J. (1988). Preschool children can learn to transfer: Learning to learn and learning from example. *Cognitive Psychology, 20,* 493-523.

Bransford, J. D., Brown, A. L., & Cocking, R.R. (1999). *How people learn: Brain, mind, experience, and school.* National Research Council, Commission on Behavioral & Social Sciences & Education. Committee on Developments in the Science of Learning. Washington, DC: National Academy Press.

Carpenter, T., Fennema, E.,; Fuson, K, Hiebert, J. Human, P., Murray, H., Olivier, A., & Wearne, D. (1999). Learning basic number concepts and skills as problem solving. In E. Fennema & T. Romberg (Eds.) ,*Mathematics classrooms that promote understanding* (pp. 45-61). Mahwah, NJ: Erlbaum.

Dunbar, K. (1995). How scientists really reason: Scientific reasoning in real-world laboratories. In R. J. Sternberg &J. E. Davidson (Eds),. *The nature of insight* (pp. 365-395) Cambridge, MA: MIT Press.

Dunbar, K. (1997). How scientists think: On-line creativity and conceptual change in science. In T. B. Ward, S. M. Smith, & J. Vaid (Eds.), *Creative thought: An investigation of conceptual structures and processes* (pp. 461-493). Washington, D.C.: American Psychological Association.

Dunbar, K. (2001). The analogical paradox: Why analogy is so easy in naturalistic settings yet so difficult in the psychological laboratory. In D. Gentner, K. J. Holyoak, & B. N. Kokinov (Eds.), *The analogical mind: Perspectives from cognitive science* (pp. 313-334). Cambridge, MA: MIT Press.

Gentner, D. (1983). Structure-mapping: a theoretical framework for analogy. *Cognitive Science, 7,* 155-170.

Gentner, D., & Toupin, C. (1986). Systematicity and surface similarity in the development of analogy. *Cognitive Science*, 10, 277-300.

Gentner, D., Holyoak,, K. J.& Kokinov, B. (2001). *The analogical mind: Perspectives from cognitive science.* Cambridge, MA: MIT Press.

Gentner, D. (1983). Structure-mapping: a theoretical framework for analogy. *Cognitive Science, 7,* 155-170.

Gentner, D., & Holyoak, K. J. (1997). Reasoning and learning by analogy: Introduction. *American Psychologist, 52,* 32-24.

Gick, M.L.,& Holyoak, K. J. (1980). Analogical problem solving. *Cognitive Psychology, 15,* 306-355.

Gick, M.L. & Holyoak, K. J. (1983). Schema induction and analogical transfer. *Cognitive Psychology, 15*, 1-38.

Holyoak & Thagard (1989). Analogical mapping by constraint satisfaction. *Cognitive Science, 13,* 332-340.

Holyoak, K., Novick, L., Melz, E.R. (1994). Component processes in analogical transfer: Mapping, pattern completion, and adaptation. In *2.* K. J. Holyoak & J. A.

Barnden (Eds.), *Advances in connectionist and neural computation theory, Vol 2: Analogical* connections (pp. 113-180). Norwood, NJ: Ablex.

Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *Elementary School Journal.* 102, 59-80.

Novick, L. (1988) Analogical transfer, problem similarity, and expertise. *Journal of Experimental psychology: Learning, Memory, and Cognition, 14*, 510-520.

Novick, L., & Holyoak, K. J. (1991). Mathematical problem solving by analogy. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 17*, 398-415.

Reed, S. K., & Demptster, A., & Ettinger, M. (1985). Usefulness of analogous solutions for solving algebra word problems. *Journal of experimental psychology: Learning, Memory and Cognition, 11,* 106-125.

Reed, S. K. (1987). A structure-mapping model for word problems. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 13*, 124-139.

Ross, B. (1987). This is like that: the use of earlier problems and the separation of similarity effects. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 13,* 629-639.

California Standards Testing and Reporting Program (STAR) (2001). Stanford 9 and California Achievement Test results. http://star.cde.ca.gov/star2001/

Stigler, J.W., Gonzales, P., Kawanaka, T., Knoll, S., & Serrano, A. (1998). *Methods and findings of the TIMSS videotape classroom study*. Washington, DC: US Government Printing Office.