Analysis of the Dynamics of Reasoning Using Multiple Representations

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Abstract

This paper presents a formalisation and analysis method for the dynamics of a reasoning process in which multiple representations play a role. Dynamics of reasoning processes are described by reasoning traces consisting of sequences of reasoning states over time. Reasoning states have a compositional structure; they are composed of different parts, for example, for different representations. Transitions between two reasoning states model reasoning steps. In relation to the compositional structure of the states, transitions are classified into a number of types. An example reasoning process involving multiple representations is used to illustrate how its dynamics can be formalised and analysed using the approach.

Introduction

Within Cognitive Science in recent years the dynamical perspective on cognitive phenomena has been emphasized and received much attention. In most literature focussing on the dynamics of cognition, the Dynamical Systems Theory (DST) is taken as a point of departure; e.g., (Port and Gelder, 1995). This theory assumes that, in contrast to the use of symbolic representations, modelling and analysis of dynamics of cognitive phenomena can be done more effectively by using representations based on real numbers and mathematical techniques, in particular difference and differential equations. The convincing examples illustrating the usefulness of this perspective often address lower level cognitive processes such as sensory or motor processing. Indeed one of the advantages of the Dynamical Systems Theory is that it is able to model the temporal aspects of events taking place on a continuous time scale, such as, for example, recognition time, response time, and time involved in motor patterns and locomotion.

Also some examples of higher level cognitive processes have been addressed using DST; for example the dynamic models for decision making developed by Busemeyer and Townsend (1993). Especially the continuous adaptive aspects of the decision making are covered nicely in this approach. Areas for which the quantitative approach based on DST is assumed to have less to offer are the dynamics of higher level processes with mainly a qualitative character, such as certain capabilities of language processing and reasoning. In the last two decades, within the areas of Computer Science and Artificial Intelligence alternative techniques have been developed to analyse the dynamics of phenomena using qualitative means. Examples are process algebra; transition systems; dynamic and temporal logic; event, situation and fluent calculus; e.g., (Eck, et al. 2001; Hölldobler and Tielscher, 1990; Kowalski and Sergot, 1986; Reiter, 2001). Just as difference or differential equations, these alternative techniques allow to consider and relate states of a process at different points in time. The form in which these relations are expressed can cover both quantitative and non-quantitative aspects. This paper illustrates the usefulness of such an approach for the analysis and formalisation of the dynamics of reasoning. Here a broad perspective is taken on reasoning, subsuming, for example, reasoning involving multiple representations.

A formal analysis method for the dynamics of reasoning is presented and illustrated by an example reasoning pattern involving geometric and arithmetic representations. This pattern is analysed and characterised in terms of a set of dynamic properties. The properties have been formalized, thus enabling automated support of analysis by an analysis environment that has been developed.

Below, first the dynamic perspective on reasoning is discussed in some more detail. Next, the example reasoning pattern is introduced, and the first steps of an analysis are made. Third, a number of dynamic properties identified for the example reasoning pattern are presented. Finally the analysis method is summarised and the contribution of the research presented in the paper is discussed.

Reasoning Dynamics

Analysis of the cognitive capability to perform reasoning has been addressed from different areas and angles. Within Cognitive Science, the two dominant streams are the syntactic approach (based on inference rules applied to syntactic expressions, as common in logic), e.g., (Rips, 1994), and the semantic approach (based on construction of mental models); e.g., (Johnson-Laird, 1983; Yang and Johnson-Laird, 1999).

Reasoning steps in natural contexts are usually not restricted to the application of logical inference rules. For example, a step in a reasoning process may involve translation of information from one representation form (e.g., geometrical) into another one (e.g., arithmetical). Or, an additional assumption can be made, thus using a dynamic set of premises within the reasoning process. Decisions made at specific points in time during the process, for example, on which representations to use or which assumptions to make, are an inherent part of the reasoning. Such reasoning processes or their outcomes cannot be

understood, justified or explained without taking into account these dynamic aspects.

To formalise the dynamics of a reasoning process, traces are used. *Reasoning traces* are time-indexed sequences of *reasoning states* over a time frame; for stepwise reasoning processes the set of natural numbers as a time frame is an appropriate choice. The set of all possible reasoning states defines the space where the reasoning takes place. Reasoning traces can be viewed as trajectories in this space, for which every (reasoning) step from one reasoning state to the next one is based on an *allowed transition*. If the possible reasoning states and the allowed reasoning steps or transitions are characterised, the set of proper reasoning traces can be defined as the set of all possible sequences of reasoning states consisting only of allowed transitions.

Reasoning States

A *reasoning state* formalises an intermediate state of a reasoning process. The content of such a reasoning state usually can be analysed according to different aspects or dimensions. For example part of the state may contain a geometric representation, another part an arithmetic representation. Accordingly, the reasoning state is structured as a composition of (i.e., a tuple of) a number of parts, indexed by some set I. This index set includes different aspects or views taken on the state, e.g., I = {geometric, arithmetic}. The set of reasoning states RS can be characterised as a Cartesian product $RS = \prod_{i \in I} RS_i$ where RS_i is the set of all states for the aspect indicated by i. For example, $RS_{geometric}$ may denote the set of all possible geometric representations; note, however, that is also possible to use more dimensions, e.g., different types of geometric representations can be formalised. This Cartesian product formalises the multi-dimensional space where the reasoning takes place. For a reasoning state, which is a vector $S = (S_i)_{i \in I} \in RS$ in this space, the S_i are called its *components*.

Reasoning Steps: Transitions of Reasoning States

A transition from one reasoning state to another reasoning state, i.e., an element $< S, S' >$ of $RS \times RS$, formalises one *reasoning step*; sometimes also denoted by $S \rightarrow S'$. A *reasoning transition relation* is a relation on $RS \times RS$. Such a relation can be used to specify the *allowed transitions*. Transitions differ in the set of components that are involved. The most complex transitions change all components of the state in one step. However, within stepwise reasoning processes, usually transitions only involve a limited number of components of the state, e.g., only one or two. Transitions can be classified according to which set of components is involved. The most simple types of transition involve a single component transition. Next come transition types where two components are involved. In the current approach we concentrate on these two classes of transition types.

Single component transition types
For example, when a modification in the reasoning state is made solely within a geometric representation, only the geometric component of the state changes (geometric reasoning step). Or, if a calculation (arithmetic reasoning) step is performed, only the aritmetic component is changing. These single component transitions involve only that component and can be defined within one component only:

$$\text{geometric} \rightarrow \text{geometric}$$
$$\text{arithmetic} \rightarrow \text{arithmetic}$$

It is also possible that one component of a state is changed by information acquisition, importing information from an external source in the reasoning process.

Transitions involving two components of a reasoning state
Other types of transitions involve more than one component. For example, if information from a geometric representation is translated into an arithmetic form, thereby extending the arithmetic representation, then two components of the state are involved: the arithmetic component and the geometric component. Examples of transition types involving two components are:

$$\text{geometric x arithmetic} \rightarrow \text{geometric}$$

(e.g., the geometric representation is extended or modified with results from the arithmetical representation)

$$\text{arithmetic x geometric} \rightarrow \text{arithmetic}$$

(e.g., the arithmetic representation is extended or modified with results from the geometrical representation)

Reasoning Traces

Reasoning dynamics results from successive reasoning steps, i.e., successive transitions from one reasoning state to another. Thus a *reasoning trace* is constructed: a time-indexed sequence of reasoning states $(\gamma_t)_{t \in T}$, where T is the time frame used (the natural numbers). A reasoning trace can be viewed as a trajectory in the multi-dimensional space $RS = \prod_{i \in I} RS_i$ of reasoning states. An example of such a reasoning trace will be discussed in Section 3; see also Figure 1. Reasoning traces are sequences of reasoning states subject to the constraint that each pair of successive reasoning states in this trace forms an allowed transition. A trace formalises one specific line of reasoning.

Example Reasoning Process

An example multi-representation reasoning process is used to illustrate the approach put forward: interaction between arithmetical reasoning and geometrical reasoning. The example focuses on how to determine the outcome of multiplications such as 23×36. Experiences on using such processes with children (8-9 years old) in class rooms have been reported, e.g., by Dekker et al. (1982), see also (Hutton, 1977). The example can also be extended to an example for children of 13 or 14 years to support algebra by geometric visualisations, e.g., the algebraic identity $(a + b)^2 = a^2 + 2ab + b^2$ interpreted as the area of a partitioned square of $(a + b) \times (a + b)$ in relation to areas of its parts: a square of $a \times a$, a square of $b \times b$, and two rectangles of $a \times b$. Also teaching quadratic equations can be supported by such visualisations as discussed, e.g., by Bruner (1968), pp. 59-63. The example pattern shows two types of one component transitions of reasoning states, and two transition types involving two components:

- an *arithmetical* reasoning step: arithmetic → arithmetic
- a *geometrical* reasoning step: geometric → geometric
- a *translation* of an arithmetical representation into a geometrical representation: geometric x arithmetic → geometric
- a *translation* of a geometrical representation into an arithmetical representation: arithmetic x geometric → arithmetic

The idea is that only simple arithmetical steps are required. The more complicated steps are performed via the geometrical representation. A number of skills are assumed. These skills can be defined in the form of transitions.

A. *Assumed arithmetic skills* arithmetic → arithmetic
aa1. splitting a number in 'tens' and single digits: 28 = 20 + 8
aa2. addition of a list of numbers of up to 4 digits, such as 1200 + 340 + 120 + 6
aa3. multiplication of two numbers starting with a nonzero digit, followed by zero or more zeros, such as 20 x 8, 60 x 30.

B. *Assumed geometric skills* geometric → geometric
gg1. partitioning a rectangle in non-overlapping areas based on partitionings of its sides
gg2. determining the area of a figure from the areas of a (non-overlapping) partition

C. *Assumed translation skills*
geometric x arithmetic → geometric:
ag1. drawing a rectangle with arithmetically given dimensions
ag2. partitioning a line segment according to a splitting of its length
ag3. determining the area of a rectangle from the multiplication of the lengths of its sides
arithmetic x geometric → arithmetic:
ga1. translating the area of a rectangle into the multiplication of the lengths of its sides
ga2. translating the area of a combination of nonoverlapping areas into the sum of the areas
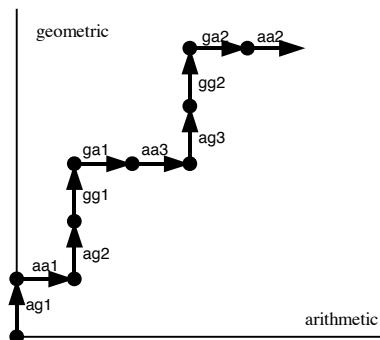
Figure 1: Reasoning trace as a trajectory in a two-dimensional reasoning state space.

The example reasoning trace, based on class room observation (cf. Dekker et al., 1982), forms a trajectory in the two-dimensional reasoning state space

$$RS = RS_{arithmetic} \times RS_{geometric}$$

This trajectory is depicted in Figure 1. Note that in this Figure only the changing component is visualised by an arrow, not what component affected this change. Therefore, e.g., both a geometric reasoning step and a translation of an arithmetic into a geometric representation are depicted by a vertical arrow. The detailed trace is presented below.

Starting problem   What is the outcome of the multiplication 23 x 36 ?

Step 1 ag1 representation translation
Create a rectangle of 23 x 36.

Step 2 aa1 arithmetic reasoning
Split the numbers into the 'tens' and single digits: 23 = 20 + 3; 36 = 30 + 6
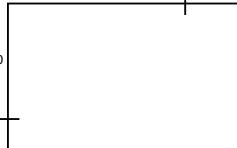
Step 3 ag2 representation translation
Translation of the arithmetical splitting of the numbers into partitions of the sides within the geometrical representation.
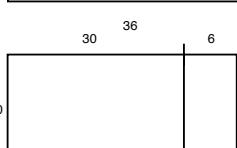
Step 4 gg1 geometric reasoning
Partition the area of the rectangle according to the partitioning of the sides.
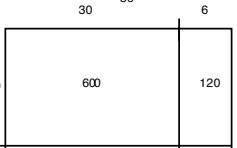
Step 5 ga1 representation translation
For each part identify the correspond-ing arithmetical expression for its area: 20 x 30, 20 x 6, 3 x 30, 3 x 6

Step 6 aa3 arithmetic reasoning
Determine the outcomes of the four multiplications 20 x 30 = 600; 20 x 6 = 120; 3 x 30 = 90; 3 x 6 = 18
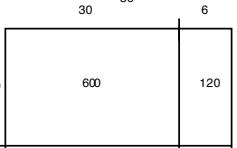
Step 7 ag3 representation translation
Identify the areas of the parts of the rectangle based on the outcomes of the multiplications.
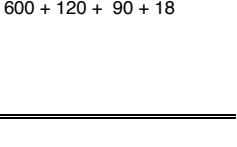
Step 8 gg2 geometric reasoning
Assert that the area of the rectangle as a whole is the combination of the areas of the parts
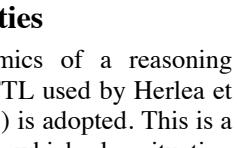
Step 9 ga2 representation translation
Identify the corresponding arithmetical relation: 600 + 120 + 90 + 18

Step 10 aa2 arithmetic reasoning
Calculate the sum: 600 + 120 + 90 + 18 = 828
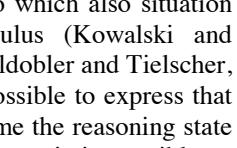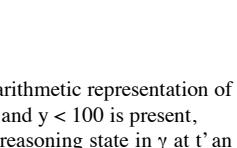
Dynamic Properties

To specify properties on the dynamics of a reasoning process, the temporal trace language TTL used by Herlea et al. (1999), and Jonker and Treur (1998) is adopted. This is a language in the family of languages to which also situation calculus (Reiter, 2001), event calculus (Kowalski and Sergot, 1986), and fluent calculus (Hölldobler and Tielscher, 1990) belong. In short, in TTL it is possible to express that in a given trace at a certain point in time the reasoning state has a certain (state) property. Moreover, it is possible to relate such state properties at different points in time. As an example, the following (global) property of a reasoning trace γ is considered, which expresses that all multiplication problems in two digits eventually will be solved.

GP1
at any point in time t
if   in the reasoning state in trace γ at t an arithmetic representation of a multiplication problem for numbers x and y < 100 is present,
then   a time point t'≥t exists such that in the reasoning state in γ at t' an arithmetic representation of a solution z of this multiplication problem with z = x∗y is included.

The formalisation of this property in TTL is as follows.

$\forall t \; \forall x, y < 100 \; \text{state}(\gamma, t, \text{arithmetic}) \models \text{multiplication\_problem}(x, y)$
$\Rightarrow \exists t' \geq t \; \exists z \quad z = x*y \; \&$
$\text{state}(\gamma, t', \text{arithmetic}) \models \text{is\_solution\_for\_multiplication\_of}(z, x, y)$

Note that for simplicity no maximal allowed response time has been specified. If desired, this can be simply added by putting a condition $t' \leq r$ in the consequent with r the maximal response time.

Milestone Properties

Within the overall reasoning process a number of milestones can be defined, and properties can be identified that express whether the process from one milestone to another one has been performed properly. Apart from the start and the finish, two intermediate milestones were defined: a reasoning state in which the problem has been represented in a geometric representation and it has been decomposed geometrically (after step 4 in the example trace), and a reasoning state in which a geometric representation with numbers in the areas occurs, i.e., in which the subproblems have been solved (after step 7 in the example trace). Accordingly, the following milestone properties have been formulated.

MP1

at any point in time t
if   in the reasoning state in trace γ at t an arithmetic representation of a multiplication problem for numbers x and y < 100 is present,
then   a time point $t' \geq t$ exists such that in the reasoning state in γ at t' a geometric representation of a rectangle ABCD is included with points P on AB and Q on AD, with |AB| = x and |AD| = y
and   this rectangle is partitioned into four areas A_{11}, A_{12}, A_{21}, A_{22} by two lines PP'//AD and QQ'//AB with P' on CD and Q' on BC with |AP| = x_1, |PB| = x_2, |AQ| = y_1, and |QD| = y_2, where x_1, y_1 is the 10-part of x, resp. y, and x_2, y_2 is the digit part of x, resp. y.

Here, |AB| is the length of AB, and // is 'in parallel with'.

MP2

at any point in time t
if   in the reasoning state in trace γ at t a geometric representation of a rectangle ABCD is included with points P on AB and Q on AD, with |AB| = x and |AD| = y,
and   this rectangle is partitioned into four areas A_{11}, A_{12}, A_{21}, A_{22} by two lines PP'//AD and QQ'//AB with P' on CD and Q' on BC with |AP| = x_1, |PB| = x_2, |AQ| = y_1, and |QD| = y_2, where x_1, y_1 is the 10-part of x, resp. y, and x_2, y_2 is the digit part of x, resp. y,
then   a time point $t' \geq t$ exists such that in the reasoning state in γ at t' in each of these areas Aij a number zij is represented which equals x_i*y_j.

MP3

at any point in time t
if   in the reasoning state in trace γ at t a geometric representation of a rectangle ABCD is included with |AB| = x and |AD| = y
and   this rectangle is partitioned into four nonoverlapping rectangle areas A_{11}, A_{12}, A_{21}, A_{22},
and   in each of these areas A_{ij} a number zij is represented which equals x_i*y_j, where x = x_1 + x_2, and y = y_1 + y_2,
then   a time point $t' \geq t$ exists such that in the reasoning state in γ at t' an arithmetic representation of a solution z with z = x*y of the multiplication problem (x, y) is included.

Local Properties

In this section a number of properties are identified that characterise the reasoning in a more local manner: each property characterises one reasoning step. For the sake of simplicity, for the example reasoning process persistence of representations in reasoning states over time is assumed, so that persistence does not need to be formulated within each of the properties.

LP1 (arithmetic-geometric)

at any point in time t
if   in the reasoning state in trace γ at t an arithmetic representation of a multiplication problem for numbers x and y < 100 is present,
then   a time point $t' \geq t$ exists such that in the reasoning state in γ at t' a geometric representation of a rectangle ABCD with |AB| = x and |AD| = y is included.

This dynamic property expresses that in reasoning trace γ, if an arithmetically represented multiplication problem occurs, this eventually is translated into a geometric representation. The formalisation of this property in TTL is as follows.

$\forall t \; \forall x, y < 100 \; \text{state}(\gamma, t, \text{arithmetic}) \models \text{multiplication\_problem}(x, y)$
$\Rightarrow \exists t' \geq t \; \exists A, B, C, D$
$\text{state}(\gamma, t', \text{geometric}) \models \text{rectangle}(A, B, C, D) \; \& \; |AB| = x \; \& \; |AD| = y$

Further local properties are the following (not in any particular order).

LP2 (arithmetic-arithmetic)

at any point in time t
if   in the reasoning state in trace γ at t an arithmetic representation of a multiplication problem for numbers x and y < 100 is present,
then   a time point $t' \geq t$ exists such that in the reasoning state in γ at t' an arithmetic representation of a splitting of the numbers x and y in 'tens' and digits occurs, i.e., x = x_1 + x_2, y = y_1 + y_2 with x_1, y_1 multiples of 10 and x_2, y_2 <10.

LP3 (arithmetic-arithmetic)

at any point in time t
if   the reasoning state in trace γ at t contains an arithmetic representation of a multiplication problem for (x, y), with x, y multiple of 10 or less than 10,
then   a time point $t' \geq t$ exists such that in the reasoning state in γ at t' an arithmetic representation of a solution z with z = x*y for this multiplication problem for (x, y) is included.

LP4 (arithmetic-arithmetic)

at any point in time t
if   in the reasoning state in trace γ at t an arithmetic representation of an addition problem for a finite list z_1 ,..., z_n of numbers of up to 4 digits is included,
then   a time point $t' \geq t$ exists such that in the reasoning state in γ at t' a solution z = $\Sigma_{1 \leq i \leq n} z_i$ of the addition problem is included.

LP5 (arithmetic-geometric)

at any point in time t
if   in the reasoning state in trace γ at t an arithmetic representation of a splitting of the numbers x and y occurs, i.e.,
x = x_1 + x_2, y = y_1 + y_2,
then   a time point $t' \geq t$ exists such that in the reasoning state in γ at t' a geometric representation of a rectangle ABCD with |AB| = x and |AD| = y is included with points P on AB and Q on AD such that |AP| = x_1, |PB| = x_2, |AQ| = y_1, and |QD| = y_2.

LP6 (geometric-geometric)

at any point in time t

if   in the reasoning state in trace γ at t a geometric representation of a rectangle ABCD is included with points P on AB and Q on AD,

then   a time point t'≥t exists such that in the reasoning state in γ at t' the rectangle ABCD is partitioned into four areas A_{11}, A_{12}, A_{21}, A_{22} by two lines PP'//AD and QQ'//AB with P' on CD and Q' on BC.

LP7 (geometric-geometric)

at any point in time t

if   in the reasoning state in trace γ at t a geometric representation of a rectangle ABCD is included that is partioned into a number of nonoverlapping areas A_1, …, A_n,

then   a time point t'≥t exists such that in the reasoning state in γ at t' it is asserted that the area of ABCD is the combination of the areas A_1, …, A_n.

LP8 (geometric-arithmetic)

at any point in time t

if   in the reasoning state in trace γ at t a geometric representation of a rectangle ABCD with |AB| = x and |AD| = y is included with points P on AB and Q on AD such that |AP| = x_1, |PB| = x_2, |AQ| = y_1, and |QD| = y_2,

and   this rectangle is partioned into four areas A_{11}, A_{12}, A_{21}, A_{22} by two lines PP'//AD and QQ'//AB with P' on CD and Q' on BC,

then   a time point t'≥t exists such that in the reasoning state in γ at t' arithmetic representations of multiplication problems for (x_1, y_1), (x_1, y_2), (x_2, y_1), and (x_2, y_2) are included.

LP9 (geometric&arithmetic-geometric)

at any point in time t

if   in the reasoning state in trace γ at t a geometric representation of a rectangle ABCD is included with points P on AB and Q on AD,

and   this rectangle is partioned into four areas A_{11}, A_{12}, A_{21}, A_{22} by two lines PP'//AD and QQ'//AB with P' on CD and Q' on BC,

and   arithmetic representations of solutions z_{11}, z_{12}, z_{21}, z_{22} for the multiplication problems for (|AP|, |AQ|), (|AP|, |QD|), (|PB|, |AQ|), and (|PB|, |QD|) are included.

then   a time point t'≥t exists such that in the reasoning state in γ at t' within the geometric representation in each area A_{ij}, the number z_{ij} is represented.

LP10 (geometric-arithmetic)

at any point in time t

if   in the reasoning state in trace γ at t a geometric representation of a rectangle ABCD is included which is partioned into a number of areas A_1, …, A_n,

and   within each of these areas A_i a number z_i is represented,

then   a time point t'≥t exists such that in the reasoning state in γ at t' an arithmetic representation of an addition problem for z_1, …, z_n is included.

LP11 (geometric& arithmetic-arithmetic)

at any point in time t

if   in the reasoning state in trace γ at t a geometric representation of a rectangle ABCD is included with |AB| = x and |AD| = y that is partitioned into a number of nonoverlapping areas A_1, …, A_n,

and   within each of these areas A_i the number zi is represented,

and   an arithmetic representation of a solution z of the addition problem for z_1, …, z_n is included,

then   a time point t'≥t exists such that in the reasoning state in γ at t' an arithmetic representation of a solution z with z = x∗y of the multiplication problem (x, y) is included.

Relationships Between the Dynamic Properties

A number of logical relationships have been established between the properties above. First of all, the three milestone properties together imply the global property:

$$\textbf{MP1 \& MP2 \& MP3} \;\Rightarrow\; \textbf{GP1} \qquad (0)$$

Next, each of these milestone properties is implied by a number of local properties:

$$\textbf{LP1 \& LP2 \& LP5 \& LP6} \;\Rightarrow\; \textbf{MP1} \qquad (1)$$
$$\textbf{LP3 \& LP8 \& LP9} \;\Rightarrow\; \textbf{MP2} \qquad (2)$$
$$\textbf{LP4 \& LP7 \& LP10 \& LP11} \;\Rightarrow\; \textbf{MP3} \qquad (3)$$

These logical relationships, which can be depicted as an AND-tree, are helpful in the analysis of errors within a given reasoning trace. First it can be checked whether **GP1** holds. If this global property does not hold, the three properties **MP1**, **MP2**, **MP3** can be checked. Given the logical relationship (0), at least one of them will be found not to hold. This pinpoints the cause of the error in part of the process, say **MP3**. Next, (only) the local properties relating to **MP3** are checked, i.e, **LP4**, **LP7**, **LP10**, **LP11**. Again, due to (3) one of them will be found not to hold. This localises the error.

The Dynamic Analysis Method

The analysis method for the dynamics of reasoning processes as presented here is summarised as follows.

1. Identify the different dimensions or *components* of reasoning states.

2. Determine the different *types of transitions*.

3. Identify relevant *dynamic properties* for the reasoning
   a. for the process as a whole (global properties)
   b. for milestones within the process
   c. for reasoning steps (local properties)

4. Determine logical *relationships* between the different dynamic properties, in an AND-tree form; e.g.,
   a. local properties imply a milestone property, and
   b. milestone properties imply a global property.

5. For a given reasoning trace, *check* which of the dynamic properties hold and which do not hold. This can take the form of a diagnosis following the tree structure of the relationships between the dynamic properties. A software environment is available to support this checking process.

The dynamic properties identified can be of different types. Some may be assumed to hold for all proper reasoning traces, others may be used to distinguish different types of reasoning traces or reasoners.

Discussion

The analysis method for the dynamics of reasoning processes put forward and illustrated in this paper was validated on the basis of reports from experiments with 8-9 year old children in classrooms in the Netherlands (Dekker et al., 1982); a similar report has been made by Hutton

(1977). This paper shows how an analysis of these dynamics can be made using traces consisting of sequences of reasoning states over time to describe reasoning processes. It is shown for the example reasoning pattern, how characterising dynamic properties can be identified.

The language used to express dynamic allows for precise specification of these dynamic properties, covering both qualitative and quantitative aspects of states and their temporal relations. Moreover, software tools have been developed to (1) support specification of dynamic properties, and (2) automatically check specified dynamic properties against example traces to find out whether the properties hold for the traces. This provides a useful supporting software environment to evaluate empirical data on the dynamics of reasoning processes.

The same analytic method and software tools can also be applied to reasoning traces produced by software simulation models. This applicability supports the comparison of human reasoning with simulated reasoning.

Further experiments will be conducted, in which also a focus is on the control of the reasoning. For example, at what point in time a translation to a geometric representation is made, and why at that point in time? In the analysis the notion of reasoning strategy will be addressed. Due to the compositional structure of reasoning state a reasoning state can be extended with a component for control information.

Future research will also address the analysis of the dynamics of other types of practical reasoning, both from the syntactical and semantical stream, or their combination; e.g., (Johnson-Laird, 1983; Yang and Johnson-Laird, 1999; Yang and Bringsjord, 2001); see also (Stenning and Lambalgen, 2001). One component of the reasoning state may contain a syntactic formula structure, and another component a mental model or set of mental models. For example, a single component transition can be defined within a syntactic component including A and $A \rightarrow B$, for the derivation of B (and hence adding it to the component) based on the inference rule modus ponens. Yet another example, within a semantic component is a transition of a set of mental models, thus providing a formalisation of the dynamics of reasoning based on mental models.

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