# Representation Strength Influences Strategy Use and Strategy Discovery

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## Abstract

When attempting to solve a problem, individuals may activate multiple potential representations for that problem. Further, different representations may be activated more or less strongly. This study investigated how strength of problem representations is related to patterns of strategy use and strategy discovery. We hypothesized that the more strongly a particular representation is cued, the more likely participants should be to use a strategy that corresponds with that representation. Further, for individuals who do not initially have a corresponding strategy in their repertoires, the more strongly a particular representation is cued, the more likely participants should be to discover a strategy that corresponds with that representation. These hypotheses were investigated among adults solving word problems about constant change. The problems could be represented in terms of discrete change or continuous change. We varied two types of cues to discrete and continuous problem representations: linguistic cues and graphical cues. Both linguistic and graphical cues influenced strategy use, and the effects of the two cue types were additive. Among participants who did not use a continuous strategy at the outset of the study, discovery of a continuous strategy was relatively rare, and only participants who received a continuous graph tended to discover a continuous strategy. The findings suggest that it may be fruitful to consider problem representations as graded and variable rather than all-or-none.

## Introduction

One step in the process of solving a problem is forming a mental representation of important features of that problem. Problem representations have been invoked to explain many aspects of people's problem-solving behavior, including success, solution times, strategies, and errors (e.g., Kotovsky, Hayes, & Simon, 1985; Lovett & Schunn, 1998). In the present study, we investigate links between problem representations and patterns of strategy use and strategy discovery.

Problem representations are sometimes conceptualized as integrated wholes, such that a particular representation is retrieved in its entirety from memory, and applied to the problem at hand (e.g., Larkin, 1983). Although this characterization may apply in some cases (e.g., for well-practiced problems), we suggest that in most cases, problem representations are constructed at the moment of solving, based on both perceivable features of the problem and on knowledge retrieved from memory about problem content or about particular problem-solving strategies (McNeil & Alibali, 2000). We further suggest that the knowledge activated in constructing a problem representation may be more or less strongly activated, and thus, aspects of the representation may be graded rather than all-or-none (see Munakata, McClelland, Johnson, & Siegler, 1997, for discussion).

There is some support in the literature for the notion that problem representations may be graded. Kaplan and Simon (1990) studied this issue in the context of the *mutilated checkerboard* problem. In this problem, the squares from two diagonally opposite corners of a checkerboard are removed, and the solver's task is to cover the remainder of the checkerboard with dominoes, each of which covers exactly two squares, or to prove that such a covering is impossible. Because the two diagonally opposite corners of a checkerboard are the same color (both black or both white), the covering task is indeed impossible; however, this fact is notoriously difficult for solvers to discover. In their experiment, Kaplan and Simon varied the strength of various cues to the "paired-ness," or parity, of the squares. Solvers were quicker to discover that the covering was impossible when adjacent squares were labeled "bread" and "butter" than when the squares were not labeled, or when they were labeled with terms that did not form a strongly associated pair ("black" and "pink"). The bread-and-butter cue to parity facilitated a stronger representation of this crucial problem feature, and this led to faster discovery of the problem solution.

The purpose of the present study was to investigate whether variations in the strength of problem representations can account for variations across solvers in patterns of strategy use. Several past studies have investigated the links between problem representation and strategy use (e.g., Alibali, Bassok, Solomon, Syc, & Goldin-Meadow, 1999; Morales, Shute, & Pellegrino, 1985; Siegler, 1976). However, to date, little research has examined how the *strength* of representations relates to patterns of strategy use. We hypothesized that the more strongly a particular representation is activated, the more likely participants would be to use a strategy that corresponds with that representation. Further, for individuals who do not initially have a corresponding strategy in their repertoires, the more strongly a particular representation is activated, the more likely participants would be to discover a strategy that corresponds with that representation.

We also wished to examine the effects on strategy use of having multiple, incompatible representations that are simultaneously active. We hypothesize that the operative factor in determining which representation guides solution is the *relative* strength of a particular problem representation. Therefore, when multiple, potentially incompatible problem representations are simultaneously active, participants' performance should be more variable than when a single problem representation is active.

This study investigated these hypotheses among adults solving word problems about constant change (Bassok & Olseth, 1995). The problems could be represented in terms of either discrete, stepwise change or smooth, continuous change. The experiment varied two types of cues to discrete and continuous problem representations: linguistic cues and graphical cues. The linguistic cues were drawn from previous research on people's verbal descriptions of constant change problems (Alibali et al., 1999). The graphical cues were chosen based on previous research about graph comprehension (Zacks & Tversky, 1999), which indicated that line graphs cue representations of continuous changes in values, whereas bar graphs cue representations of discrete changes in values.

In some conditions in the present experiment, the linguistic and graphical cues converged on a single representation. In other conditions, linguistic cues alone were provided. In still other conditions, the linguistic cue pointed toward one representation and the graphical cue pointed toward the other representation. If stronger representations lead to more frequent use of a corresponding strategy, participants should use that strategy most often in the corresponding cues case, and least often in the conflicting cues case. The single-cue case should fall somewhere in the middle.

## Method

### Participants

Participants were 158 Introductory Psychology students at the University of Wisconsin—Madison. The sample included 58 males, 90 females, and 10 participants who did not disclose their gender. Most participants were either freshmen (58%) or sophomores (23%), and all had taken at least one semester of college-level mathematics. Students received extra credit points for Introductory Psychology in exchange for their participation.

### Procedure

Students were tested in a small classroom in groups of 15 to 25. They were given up to 45 minutes to complete a set of 10 story problems. They were instructed to work the problems in the order presented and not to return to earlier problems after solving later ones. They were also asked to show all of their work and to circle their final solution for each problem. Students were not permitted to use calculators.

### Materials

Students received a packet of 10 words problems about constant change, based on those used in prior studies (e.g., Alibali et al., 1999; Bassok & Olseth, 1995). The first 8 problems in each packet focused on quantities that changed continuously (e.g., rain falling, a tree growing), and these problems were the site of the manipulation. As seen in Table 1, the wording of the problems was varied to cue either a discrete representation or a continuous representation. Cues to the discrete representation included amount-like units for the initial and final quantities (e.g., 5 millimeters), mention of individual units of time (e.g., the 12 weeks), and explicit reference to the constant. Cues to the continuous representation included rate-like units for the initial and final quantities (e.g., 5 millimeters per week), mention of the entire period of time (e.g., the 12-week period), and explicit reference to rate.

In addition, as seen in Figure 1, the problems were accompanied either by bar graphs, by line graphs, or by no graphs at all. Thus, the study utilized a 2 (verbal cues: discrete or continuous) x 3 (graphs: discrete [bar], continuous [line], or none) between-subjects design. The final two problems were transfer problems that were the same across all conditions, and they utilized discrete content (e.g., plants per row in a garden), discrete wording, and no graph. Participants' performance on the transfer problems is not addressed in this paper.

Table 1: Sample Problems

| |
|---|
| *Discrete Wording* |
| A sapling grows for 12 weeks. The number of millimeters it grows in each successive week increases by a constant from the number in the previous week. In the first week the sapling grows 5 millimeters and in the twelfth week it grows 137 millimeters. How many millimeters does the sapling grow in total over the 12 weeks? |
| *Continuous Wording* |
| A sapling grows for a period of 12 weeks. The rate at which it grows increases steadily over the period, from 5 millimeters per week at the beginning of the first week to 137 millimeters per week at the end of the twelfth week. How many millimeters does the sapling grow in total over the 12-week period? |

### Coding

Each problem was initially scored as correct, incorrect, or no response. Next, the strategy that each participant used to solve each problem was coded, and all strategies were classified as either discrete, continuous, or other (unclassifiable). Coding definitions are presented in Table 2. Most strategies in the "other" category were conceptually flawed attempts to solve the problems (e.g., adding or multiplying the initial and final values, or multiplying the initial value by the number of intervals and then adding the final value).

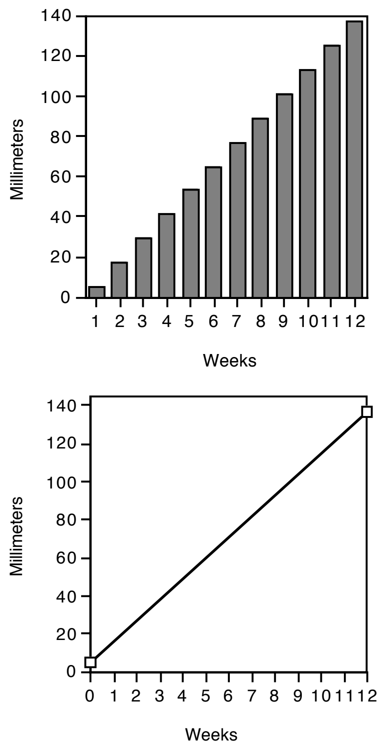



Figure 1: Sample bar and line graph.

Data from a subsample of 30 participants were rescored by a second coder to establish reliability. Agreement between the coders was 97% ($N = 282$ problems).

Table 2: Strategy Codes

| Strategy | Definition |
|---|---|
| *Discrete Strategies* | |
| Sum | Participant finds the constant increase, calculates the value for each interval, and adds these values |
| Gauss | Participant adds values for initial and final intervals and multiplies this sum by half of the number of intervals |
| *Continuous Strategies* | |
| Average | Participant finds average value per interval and multiplies by number of intervals |
| Calculus | Participant sets up equation and integrates |
| Area | Participant calculates area using geometric methods (e.g., adds areas of rectangle and triangle) |

## Results

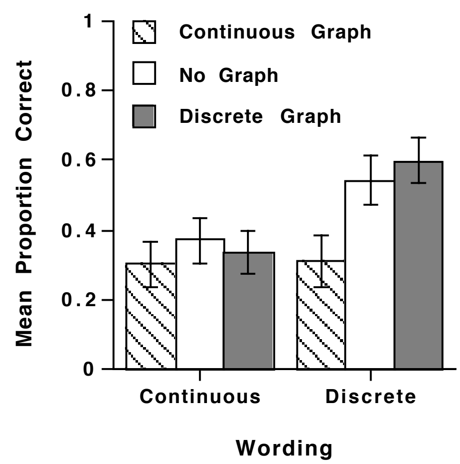
Our analysis focuses on the effects of cues to discrete and continuous representations on participants' overall level of performance, their strategy use, and their strategy discovery. The analyses reported here focus on the first eight problems, where the experimental manipulation occurred.

### Problem solutions

We first examined whether variations in problem wording and graphs influenced whether participants solved the problems correctly. The interaction of wording and graphs was not significant, but there were main effects of both factors. As seen in Figure 2, participants who received problems with discrete wording were more successful than participants who received problems with continuous wording, $F(1, 152) = 7.22$, $p < .01$, despite the fact that all of the problems involved quantities that changed in a continuous fashion. Graphs also influenced success, $F(2, 152) = 3.61$, $p < .05$. Participants who received problems with continuous graphs performed most poorly, and participants who received discrete graphs and no graphs performed similarly well. Post hoc tests indicated that the discrete-graph and no-graph groups each differed significantly from the continuous-graph group, but they did not differ from one another.

Why were continuous wording and continuous graphs associated with poorer performance? Before addressing this question, we first consider patterns of strategy use.

Figure 2: Proportion of problems solved correctly by participants in each group.




### Strategy use

Participants used discrete strategies much more often than continuous strategies in the dataset as a whole (63% vs. 16% of trials). Because of this, we used frequency of discrete strategy use rather than frequency of continuous strategy use as the dependent measure in our analysis of strategy use, to avoid possible floor effects in some of the cells of the design.

Once again, the interaction of wording and graphs was not significant, but there were main effects of both factors. As seen in Figure 3, participants who received discrete wording used discrete strategies more frequently than participants who received continuous wording, $F(1, 152) = 6.97$, $p < .01$. Graphs also influenced whether participants used discrete strategies, $F(1, 152) = 15.10$, $p < .001$. Participants who received continuous graphs used discrete strategies least often, and participants who received discrete graphs and no graphs used discrete strategies much more often. Post hoc tests indicated that the discrete-graph and no-graph groups both differed significantly from the continuous-graph group, but they did not differ from one another. Thus, both types of cues influenced participants' strategy use, and the effects of wording and graphs appear to be additive.

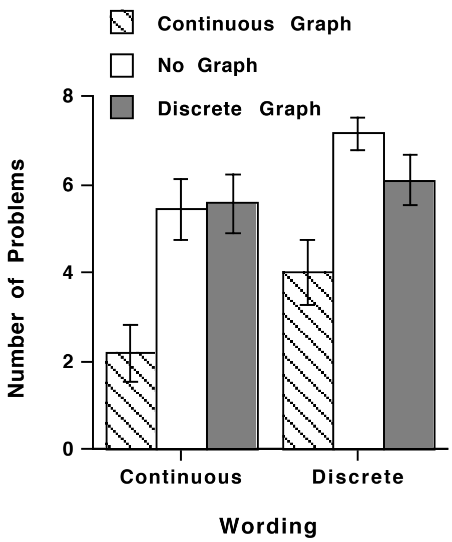



Figure 3: Number of problems solved using discrete strategies by participants in each group.

We now return to the question of why continuous wording and continuous graphs were associated with poorer problem-solving success in the initial analysis. The strategy analysis indicates that participants who received continuous wording or continuous graphs were less likely to use discrete strategies. Perhaps participants also applied discrete strategies more successfully than continuous ones, and this accounts for the performance differences seen in Figure 2. However, the success rates for discrete and continuous strategies did not differ. Participants succeeded on 50% of trials on which they used discrete strategies ($N = 1012$), and 52% of trials on which they used continuous strategies ($N = 238$).

To shed additional light on the performance differences, we examined the number of trials (out of 8) on which participants used discrete strategies, continuous strategies, or other, unclassifiable strategies, which were typically incorrect. These data are presented in Table 3. As seen in the table, participants who received continuous wording or continuous graphs used more strategies in the "Other" category than participants who received discrete wording and discrete graphs, or discrete wording and no graphs. Strategies in this category were often conceptually flawed attempts to solve the problems, suggesting that participants who received continuous wording or graphs were often at a loss as to how to solve the problems. They may have attempted various strategies in an effort to generate or discover a strategy compatible with the continuous problem representation.

Based on the data in Table 3, we next examined whether participants who received converging cues to a single representation used the corresponding strategies more often than participants who received only a single cue. Participants who received converging cues to continuous representations (i.e., wording and graphs) were indeed more likely to use continuous strategies than participants who received only a single cue (i.e., wording only), $F(1, 152) = 9.38$, $p < .01$. For discrete representations, there was no significant difference in discrete strategy use between participants who received converging cues to a discrete representation and participants who received only a single cue (indeed, the direction of the effect is opposite that predicted). This lack of difference suggests that the discrete representation was readily accessible even in the absence of the discrete graph.

Table 3: Mean Number of Strategies of Each Type Used by Participants in Each Group

| | | Strategy Category | | |
|---|---|---|---|---|
| Wording | Graph | Discrete | Contin. | Other |
| Discrete | Discrete | 6.07 | 0.74 | 1.15 |
| Discrete | Contin. | 3.96 | 2.00 | 1.46 |
| Discrete | None | 7.16 | 0.00 | 0.76 |
| Contin. | Discrete | 5.59 | 0.59 | 1.78 |
| Contin. | Contin. | 2.15 | 3.11 | 2.70 |
| Contin. | None | 5.46 | 0.92 | 1.54 |

We also compared the strategy use of participants who received converging cues to a single representation and participants who received cues to both representations. Participants who received converging cues to discrete representations used discrete strategies more often than participants who received discrete wording and continuous graphs, $F(1, 152) = 5.46$, $p = .02$, but they did not differ significantly from participants who received continuous wording and discrete graphs. Participants who received converging cues to continuous representations used continuous strategies more often than participants who received continuous wording and discrete graphs, $F(1, 152) = 12.62$, $p < .001$, but they did not differ from participants who received discrete wording and continuous graphs, although the effect was in the predicted direction, $F(1, 152) = 2.18$. This pattern of findings suggests that, for problems like those used in the present study, graphs may be more effective than wording as a cue to problem representations.

### Strategy discovery

We next turn to the issue of strategy discovery. As noted above, continuous strategies were used relatively infrequently in the dataset as a whole. However, some participants who did not use continuous strategies at the outset of the session appeared to "discover" continuous strategies in the course of solving the eight problems. Note that we do not mean to imply that these participants invented continuous strategies "from scratch." Instead, we believe that participants realized that they could apply familiar techniques such as averaging or calculating area as a method for solving the constant change problems. In this sense, they "discovered" continuous strategies.

Was discovery of a continuous strategy facilitated by cues to a continuous representation? To address this question, we eliminated all participants who used a continuous strategy on the very first problem ($N = 19$), because those participants may have already had the continuous strategy in their repertoire before the session began, instead of discovering it during the session. We then examined the proportion of remaining participants in each group who used a continuous strategy on at least one of the remaining seven problems.

As seen in Table 4, the only groups in which a substantial proportion of participants discovered the continuous strategy were those that received continuous graphs. Thus, graphs were an important cue in fostering the discovery of a continuous strategy. It is also worth noting that the continuous strategy was discovered most often in the group that received converging cues to a continuous strategy, and indeed, was never discovered in the group that received only a single cue (wording) to a continuous strategy. Similarly, the continuous strategy was never discovered among the group that received continuous wording with a discrete graph. However, 18% of participants who received discrete wording with a continuous graph discovered the continuous strategy. On the whole, the data are compatible with the view that a stronger representation is more likely to lead to strategy discovery.

Table 4: Percent of Participants in Each Group Who Discovered a Continuous Strategy

| Wording | Graph | % who Discovered Continuous Strategy |
|---|---|---|
| Discrete | Discrete | 8 |
| Discrete | Contin. | 18 |
| Discrete | None | 0 |
| Contin. | Discrete | 0 |
| Contin. | Contin. | 26 |
| Contin. | None | 0 |

## Discussion

In this study, both linguistic and graphical cues to problem representations influenced participants' strategy choices and strategy discovery. The overall analysis indicated main effects of both cue types on success and strategy use. Focused contrasts indicated that participants who received converging cues were more likely to use a target strategy than were participants who received a verbal cue to the target representation but a graphical cue to the alternative representation. For the continuous representation, participants who received converging cues were also more likely to use corresponding (continuous) strategies than participants who received the wording cue alone (i.e., with no accompanying graph). However, for the discrete representation, participants who received converging cues and participants who received wording cues alone used corresponding (discrete) strategies about equally often.

Although discovery of a continuous strategy was rare in the sample as a whole, graphical cues appeared to be especially important for strategy discovery. Continuous strategies were discovered most often in the group that received converging cues to a continuous representation, and second most often in the group that received discrete wording with a continuous graph. These findings underscore the importance of graphical representations in helping individuals construct mental models of problem situations. Our findings are compatible with Kalchman, Moss and Case's (2001) claim that line graphs are especially important in the development of understanding of mathematical functions.

Several aspects of the results converge to suggest that the "default" representation for constant change problems is one of discrete rather than continuous change. First, the large majority of problems were solved using discrete strategies. Second, participants who received bar graphs performed similarly to participants who received no graphs at all. This suggests that participants did not need the aid of the bar graphs in order to construct discrete mental models of the problem situations. They appeared to construct discrete representations spontaneously, even in the absence of the bar graphs. In contrast, participants who received line graphs performed quite differently from participants who received no graphs. The line graphs appeared to help participants construct continuous mental models of the problem situations, as the strategy discovery data suggest.

Why might discrete representations be more readily available to participants than continuous ones? One possibility has to do with the nature of the mathematical relations that are involved in working with the representations. Strategies compatible with discrete representations tend to rely on additive relations (e.g., summing the values for each increment), whereas strategies compatible with continuous representations tend to rely on multiplicative relations (e.g., multiplying the average value times the number of increments). Because additive relations are simpler and more fundamental than multiplicative ones, they may be noticed first. This hypothesis implies that individuals with strong mathematics skills should be especially likely to use continuous strategies a possibility we intend to examine in future work.

Even participants who received both linguistic and graphical cues to a continuous representation used continuous strategies relatively infrequently. The high incidence of unclassifiable strategies among participants who received continuous cues suggests that many participants were unable to generate a strategy compatible with the continuous representation. It is possible that some of these unclassifiable strategies were generated based on hybrid representations that combined both discrete and continuous elements. If the "default" representation for constant change problems is discrete, as we argued above, then cues to a continuous representation may create a situation in which multiple, incompatible representations are simultaneously active. Indeed, many of the unclassifiable strategies included both additive components, reminiscent of discrete strategies, and multiplicative components, reminiscent of continuous strategies. A more detailed analysis of these unclassifiable strategies may shed light on processes of strategy construction.

It also seems worth noting that a small number of participants altered the presented graphs. On problems with line graphs, some participants drew lines down to the x-axis to "discretize" the graph, and on problems with bar graphs, some participants drew a line across the tops of the bars to "linearize" the graph. In this regard, it is interesting to note that participants who received discrete wording and a discrete graph were slightly more likely to use continuous strategies than participants who received discrete wording and no graph. It is possible that even a bar graph can sometimes cue a continuous representation, because the linear relationship between the variables is an emergent feature of the bar graph.

In sum, the present findings add to the body of literature elucidating the links between problem representation, strategy use, and strategy discovery. For constant change problems, graphical representations, and in particular, line graphs, were important cues to strategy use and strategy discovery. However, many participants in this study failed to discover a continuous strategy. The findings suggest that people often activate multiple representations for individual problems, and if these representations are incompatible, people may have difficulty generating an effective strategy for solving the problems. More generally, the present findings suggest that, to understand patterns of strategy change and strategy discovery, it is fruitful to conceptualize problem representations as graded and variable rather than all-or-none.

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