Solving arithmetic operations: a semantic approach

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Abstract

Systematic errors observed when solving arithmetic operations are often considered as being procedural. Rules induced by the learner and the errors committed are viewed as resulting by use of general problem solving methods. In this study, in the case of solving column subtractions, we show that some errors may be semantic: they are due to analogies involving different sources that guide the interpretation of both operations and procedures already learned. Two sources have been identified: (i) subtracting considered as removing something and (ii) subtracting considered as covering a distance between two elements. Results of the 2 experiments reported here show that (i) not only all of the predicted semantic errors were observed among beginner children, but also that (ii) semantic errors were still observed among more advanced learners, in a decreasing proportion for higher level of instruction. These results support the idea that (i) semantic aspects have a major influence in the learning process, and (ii) that this kind of errors still intervene in the learning process even if the procedural aspects become more influent (advanced students). These results suggest that a procedural approach might be articulated with a semantic approach.

Introduction

Many studies show that some errors made when solving arithmetic operations (e.g. Brown & Burton, 1978; Sleeman, 1982), and notably when solving column subtractions (e.g. Young & O'Shea, 1981, VanLehn, 1982, 1983, 1987, 1990), are systematic in nature. As it has been observed, some errors are quite stable, both for within or between subjects' measures. This led researchers to reject the idea that all errors are calculation errors or due to lack of attention, and therefore, to look for mechanisms that lead to these errors.

As it has been noticed by Ohlson and Rees (1991), most of the investigators in this field focused attention on procedural mechanisms. The most prominent view for column subtractions has been developed by VanLehn and his colleagues (e.g. Brown & VanLehn, 1980; VanLehn, 1982, 1990) and has promoted the repair theory that has been implemented in the SIERRA Model (VanLehn, 1987, 1990).

In this view, conjunction of two mechanisms may lead to error production. First, learning consists on inducing rules in a syntactic way from lessons composed of solved examples, using general problem solving methods. Second, errors result from application of problem solving heuristics to overcome the impasses encountered when solving a new operation.

This theory can be qualified as procedural because the only parameters which intervene in the model are (i) knowledge about specific arithmetic facts (for instance the fact that 2 is smaller than 5 or 7 – 4 = 3), (ii) heuristics for deriving rules from previously solved examples, (iii) heuristics for solving an impasse situation which has been encountered when previously learned rules have been applied to a new situation. One characteristic of a procedural approach is that the children's interpretation of both the whole operation and the procedure is not taken into account.

Framework

Although VanLehn (e.g. 1990) shows convincing evidence of procedural errors, the exhaustiveness of his description could be questioned. In this paper, we intend to provide evidence that a procedural approach doesn't provide an explanation for some of the systematic errors observed and that articulating this approach with a semantic point of view may increase the range of explained errors. This will show as well that in a procedural situation par excellence, such as solving column subtractions that could be solved in a purely syntactical manner (Resnick, 1982), semantic aspects still influence the solving process. This point has some implications on problem solving mechanisms.

Some work focused on the influence of semantic aspects in solving column subtractions (Carpenter, Franke, Jacobs & Fennema, 1996; Fuson, 1986; Fuson & Briars, 1990; Hiebert & Wearne, 1996; Resnick, 1982; Resnick & Omanson, 1987). Through the use of analogies and/or concrete materials, these studies evaluated the influence of a teaching method and aimed on helping children to understand some of the conceptual background of column subtractions solving methods. The efficiency of these methods is rather variable. In this work, we do not adopt any position in the debate concerning virtues of providing conceptual background versus teaching procedures as if they were arbitrary, but we shall show that semantic aspects are involved spontaneously even if they are not an explicit part of the teaching method. They are expressions of a basic analogical transfer mechanism that attributes a meaning to a given situation. Such semantic influences have been identified for arithmetic operations by Fischbein (Fischbein, Deri, Nello & Marino, 1985 ; Fischbein, 1989) with the notion of tacit models. These are simple structural entities of a concrete and practical nature, which control the course of the reasoning process and are specific cases of analogy sources (Fischbein, 1987; Sander, 2000). Fischbein and his colleagues worked mainly with word problems and didn't extend their view to procedural situations such as column subtractions.

In the case of solving column subtraction, we hypothesize that errors are not necessarily due to a repair used in an impasse situation. Rather, they are sometimes a direct consequence of the interpretation of a learned procedure: the child applies the procedures according to her/his interpretation and the wrong result might be predicted from this irrelevant interpretation. In order to deal with the target situation (the column subtraction), s/he refers to a source knowledge, which is the knowledge associated with this new subtracting situation. In fact, we hypothesize that the semantic errors observed in column subtractions result from an analogical transfer in situations for which the source is non adequate. Thus, the errors are consequences of a negative transfer. We consider two sources of analogy for solving column subtractions: 'remove' and 'distance'. Both of them can be evoked spontaneously by the children or be due to the teaching method.

In the 'remove' view, subtracting is seen as taking a part out of a whole: the whole and the part are the two quantities and the result is what is left. In the 'distance' view, subtracting is seen as going from a given point to another: the departure and the arrival points are the two values and the result is the distance between them. These interpretations are valid for each value of the whole operation, but the negative transfer is due to its extension to each column and to each digit of the operation. If the operation is interpreted through these sources, the resulting errors can be predicted by these hypothesized sources. Using the terminology of VanLehn (1990), those errors are described in Table 1.

It can be noticed that most of the errors might result from distance or remove interpretations and thus, verbal reports might be useful for identifying the source. It can also be noticed that two errors (Diff 0-N=N and Diff 0-N=0) might sometimes be particular cases of other errors (respectively Smaller from Larger and Zero instead of Borrow) but they might also be specific to the cases involving zero, which could be identified either by verbal reports or by the presence of one error when the other is absent.

From our point of view, when starting to learn, children build interpretations of the operation and of the procedures through analogical transfer mechanism. This

Table 1: Definition and interpretation of semantic errors

Smaller from Larger (e.g. 457-168=311)
Definition: The smaller digit from each column is subtracted from the larger one wherever it is situated.
Remove interpretation: When a part is removed from a whole, the part is always smaller than the whole.
Distance interpretation: As a distance is symmetrical, the distance from the smaller to the larger is equal to the distance from the larger to the smaller.
Zero instead of Borrow (e.g. 457-168=300)
Definition: A zero is written instead of borrowing.
Remove interpretation: If what has to be removed is more than what is available, then it is removed and a zero is left. The impossibility of removing a quantity larger than the whole might also be marked by a zero.
Distance interpretation: In this case, distance is considered as oriented. If one considers that it is only possible to go upward, the distance is zero when the departure point is situated after the arrival point.
Blank instead of Borrow (e.g. 457-168=3__).
Definition: Nothing is written instead of borrowing.
Remove interpretation: No answer is given in the corresponding column to signal the impossibility of removing something larger than what is actually there.
Distance interpretation: The impossibility to go backward is marked by a non-answer to the corresponding column.
Stutter subtract (e.g. 457-3=124)
Definition: The last digit of the same line takes the place of a blank.
Remove interpretation: Two quantities are needed when removing a part, thus the missing quantity is replaced by the closer one.
Distance interpretation: Departure point and arrival point are both needed to go from one place to another. Thus, the missing point is replaced by the closer one.
Diff 0-N=N (e.g. 400-168=368)
Definition: If one of the upward digits is zero, the downward digit is written as the result.
Remove interpretation: Nothing can be taken from zero so the original value is unchanged.
Distance interpretation: None.
Diff 0-N=0 (e.g. 400-168=300)
Definition: If one of the upward digits is a zero, zero is written as the result.
Remove interpretation: The impossibility of taking something from zero is marked by a zero result.
Distance interpretation: None.
Diff N-0=0 (e.g. 457-100=300)
Definition: If one of the downward digits is zero, the zero is written as the result.
Remove interpretation: The impossibility of taking zero from a quantity is marked by a zero result.
Distance interpretation: None

extends the range of application of the learned procedures.

For instance, with the "remove" and "smaller from larger" interpretations of a procedure, the child will extend to "3 – 6" what has been learned from "6 – 3". S/he will consider that they are both cases of "removing a part from a whole", and that the place of the digit (either upward or downward) is not relevant because the part is necessarily the smaller number and the whole is necessarily the larger number. With acquisition of new procedures, semantic influence will persist but will decrease since specific learned procedures will narrow the extension of semantic interpretation. At the same time, procedural errors, as it has been established (e.g. VanLehn, 1990) will be developed.

Thus, we make the following hypotheses:

(i) At the beginning of learning, children will mostly make semantic errors resulting from analogical transfer with the hypothesized sources.

(ii) Semantic errors will persist even among more experienced children

(iii) The relative proportion of semantic errors of the total number of errors decreases as child level increases.

The aim of the first experiment is to test the first hypothesis and the aim of the second experiment is to test hypotheses (ii) and (iii).

Experiment 1

Subjects

Subjects were 50 grade 2 children who begun studying how to subtract but haven't began studying how to borrow. In accordance with the ministerial directives, the teaching methods in the classrooms were focusing on the procedural aspects and not on the conceptual backgrounds.

Material and procedure

Children had to solve collectively in the classroom, and without any time limit, 20 subtractions used by VanLehn (1982), which allowed identification of a large variety of errors. The instruction was: "You have to solve 20 subtractions. Do the best you can. Take all the time you need." Since 17 of the 20 required borrowing, a high rate of wrong results was expected.

This situation is quite original in the didactical field, since students are usually tested on contents that they have supposedly already studied. However, it is standard in problem solving paradigms that specific knowledge about the problems is often considered as a bias that has to be avoided. In fact, to support the existence of a general cognitive mechanism involved in this situation, we used an usual problem solving paradigm in a school situation.

Furthermore, 14 of the 50 children were randomly chosen and were tested again the next week after the first test. They were asked to solve the same 20 operations for a new test and to explain the way by which they solved them. Their verbal reports were recorded.

Results

Protocols of 8 children were excluded from the data because of their use of the borrowing procedure, probably learned at home.

Quantitative results

Despite the fact that the subjects learned only how to solve 15% of the operations (3 out of 20), they answered 84.5% of the operations in average. Only one subject answered the 3 operations corresponding to what he had actually learned.

Table 2 displays the results. Dominant errors were distinguished from partial errors, depending on their rate of occurrence within a same protocol.

Table 2: Rate of semantic errors

Error	Dominant	Partial	Total
Smaller from Larger	47.6%	19.0%	66,6%
Zero instead of Borrow	38.1%	16.7%	54.8%
Blank instead of Borrow	4.8%	2.4%	7.2%
Stutter Subtract	9.5%	11.9%	21.4%
Diff 0-N=N	50.0%	7.1%	57.1%
Diff 0-N=0	38.1%	16.7%	54.8%
Diff N-0=0	9.5%	0.0%	9.5%

As it can be noticed, some errors are very usual: 4 types of errors are observed for more than half of the subjects. Furthermore, all the hypothesized errors were observed. Few non hypothesized errors were observed, but only for a minority of the subjects (9.5%): Small-Large = Small; Small-Large = Large; N-N=N.

These results have to be contrasted with predictions of procedural approaches. As a matter of fact, VanLehn (1990), observed all those errors but did not generate a large part of them with SIERRA, that generated only "Smaller from Larger" and "Blank instead of Borrow" from this list. In other words, errors as frequent as "Zero instead of Borrow", "Diff 0-N=N" and "Diff 0-N=0" were not produced by SIERRA. All in all, 76.2% of the children revealed errors that were not predicted by this model, when only 9.5% of them revealed errors not predicted by the semantic approach.

Simulations

3 levels of simulation were performed. At the first level, each protocol was associated with a list of non competitive errors that were observed for this protocol: for instance, Diff 0-N=0 and Diff 0-N=N could not be associated with the same protocol because they lead to different results. The actual results were compared with the ones obtained when applying the identified errors to

the operation. No calculation error was allowed for explaining differences.

At the second level, calculation errors were taken into account: a difference of plus or minus 1 was accepted if this was not corresponding to another systematic error.

At the third level, all the errors observed in the protocol were taken into account even if some of them were competitive.

Thus, several results could be compatible with the same simulation and calculation errors were taken into account as well. The results are presented in Table 3.

Table 3: Simulation with semantic errors

	Sim 1	Sim 2	Sim 3
Rate of prediction	77.9%	81.3%	89.8%

Verbal reports

Verbal reports were analyzed in the following way.

First, various expressions were identified as cues for specifying a source. For instance, "I have 3 and I remove 4" was indicating a 'remove' source since 'I have' was considered as referring to a quantity and 'remove' referring to the conception of taking part of a whole. "Going from 0 to 7…" indicates a distance source because of the reference of going from one place to another. This analysis of verbal reports showed that each child was using these kinds of expressions, which support the idea that the operation is actually interpreted in terms of taking part of a whole or going from one place to another.

Second, explanations for each error were compared to the predicted interpretations. The explanations produced were consistent with the expected ones.

Leaving apart the debate concerning verbal protocols, we would like to point out that verbal reports here coincide with the expected interpretations.

Hereafter, few examples are presented.

Subject J (83-44=40; Zero instead of Borrow; remove): "There are 3. We have to remove more than what is there. It remains 0"

Subject A (1564-887=1000; Zero instead of Borrow; remove): "… I have 5 candies in my hand, I cannot take 8 out of them to eat, so I take the 5 and nothing is left…"

Subject H (6591-2697=4106; Smaller from Larger; remove): "I have 1, no I have 7, I remove 1, 6 is left; I have 9, I remove 9, 0 is left; I have 6, I remove 5, 1 is left …"

Subject D (8305-3=5002; Stutter Subtract and Diff 0-N=0; remove): "We put 5 on the fingers, we remove 3 and we notice that 2 are left. 0 can't be removed from any number so it makes 0. And 3, I remove 3, 0 is left. For 8, I remove 3 and 5 are left from the 8."

Subject F (6591-2697=4106; Smaller from Larger; distance): " We count in our head what is missing to go from a given number to the one we need. From 1 to 7, 6 is missing, thus 7 minus 1 is 6; from 9 to 9, 0 is missing; from 0 to 9, 9 is missing; from 5 to 6, 1 is missing …"

Subject B (562-3=231; Stutter Subtract & Smaller from Larger; distance). "3 minus 2 is 1 and then since there is nothing I have to use the 3. From 3 to 6, 3 is needed. From 3 to 5, 2 is needed".

Discussion

As a summary, results of this first experiment support the hypothesis that interpretative aspects are involved in solving column subtractions.

First, despite the fact that children didn't know how to solve most of the problems, they tried to answer nearly all of them. From our point of view, this result supports the idea that what the children learned was interpreted in a conceptual framework that provided solutions for new situations.

Second, all the errors predicted by the semantic perspective were actually observed, and unpredicted errors appeared only seldom, supporting the idea that they resulted from the predicted interpretations. Third, depending on the kind of simulation, the semantic perspective allowed prediction of 77.4% to 89.1% of the errors, supporting the idea that semantic errors are prominent in the beginning of learning. Fourth, the analysis of the verbal reports showed that all the children were referring to the hypothesized sources with the predicted interpretations.

Experiment 2

Subjects

409 children who had already studied subtractions with borrowing participated in this experiment. 158 were children of grade 2 and 251 were grade 3. The teaching method was the same as in experiment 1. Given that errors rate might decrease with learning, a greater number of participants was necessary in this experiment to identify the errors.

Material and procedure

The material and procedure were the same as in experiment 1. 16 participants, randomly chosen among the ones who revealed semantic errors, were selected here for verbal reports.

Results

Quantitative results

As in experiment 1, dominant errors were distinguished from partial errors. 39.4% of the children had at least one semantic error. For 13.0% of them, at least one semantic error was dominant and for the remaining 26.4%, the semantic errors were partial. The results per error are presented in Table 4. Results show that the semantic errors don't disappear with learning. They are still present, and with a great variety, for more than one third of the children, and they stay dominant for a minority of the participants.

Differences between grade 2 and grade 3 children are significant. 52.5% of the grade 2 versus 31.0% of grade 3 children had at least one semantic error (Chi2(1)=18.70; p<.01). The difference is also significant for the dominant errors: 25.9% of grade 2 versus 4.8% of grade 3 children had at least one dominant semantic error (Chi2(1)=38.52, p<.01). These differences do not reflect only the better performance of the grade 3 children but also a decrease in the proportion of semantic errors among the whole range of errors as indicated by the simulations.

Table 4: Rate of error for each semantic error

Error	Dominant	Partial	Total
Smaller from Larger	3.1%	11.3%	14.4%
Zero instead of Borrow	0.5%	5.6%	6.1%
Blank instead of Borrow	0.0%	0.0%	0.0%
Stutter Subtract	0.0%	7.1%	7.1%
Diff 0-N=N	8.7%	14.5%	23.2%
Diff 0-N=0	3.6%	16.7%	12.2%
Diff N-0=0	2.0%	6.6%	8.6%

Simulations

The method is similar to the one used in experiment 1. Table 5 displays the results of the simulations, i.e. the percentage of the errors that the semantic approach can explain. Even if there is a strong discrepancy with the beginners' simulation, the semantic perspective allows to predict within 1/4 to 1/5 of the errors for grade 2 children, depending on the kind of simulation, and about 1/7 of the third grade's errors. The differences between each level (grade 2 no borrow, grade 2 borrow, grade 3) are significant for each simulation (Fisher-Pitman Homogeneity Tests with z values from 2.74, p=.006 to 12.07, p<.0001).

Table 5: Simulation depending on participant level

	Sim 1	Sim 2	Sim 3
Grade 2 no borrow	77.9%	81.3%	89.8%
Grade 2 borrow	21.7%	22.8%	25.3%
Grade 3	14.1%	14.7%	16.3%

These results support our hypothesis that semantic errors persist even after procedural learning and that their importance decreases among the whole range of errors.

Verbal reports

Analysis of the verbal reports leads to the same conclusion as in experiment 1: all of the children referred to the predicted sources, with the predicted interpretations.

Discussion

The results of experiment 2 are consistent with hypotheses (ii) and (iii), supporting the idea that procedures are interpreted within a conceptual framework that definitely has a decreasing influence, yet does not completely disappear even after learning.

This may suggest that not only two kinds of errors exist (procedural and semantic ones) but also that the influence of one decreases when that of the other increases.

The plausibility of this suggestion is reinforced by some of VanLehn's (1990) results. Indeed, VanLehn illustrates (chapter 7) the performance of SIERRA with 33 systematic errors. SIERRA generates 25 of them. Among these 25, only 2 are considered as semantic by our approach. In contrast, among the 8 that SIERRA failed to generate, 5 are predicted by our approach. Thus, it appears that SIERRA's successes are failures of the semantic approach and vice-versa. This supports the idea of different mechanisms.

General Discussion

In this study, we showed through several converging measures that semantic aspects were involved in a procedural task par excellence: solving column subtractions.

Results are consistent with the idea that interpretative aspects should be taken into account in problem solving situations, as it has already been demonstrated in a body of research with some mathematical word problems (e.g. Bassok & Olseth, 1995; Bassok, Wu, & Olseth, 1995), with puzzle problems (Clément & Richard, 1997; Zamani & Richard, 2000), and in learning devices (Sander & Clément, 1997; Sander & Richard, 1997).

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